HOMEWORK 4, DUE WEDNESDAY FEBRUARY 8

Problem 1 (5 points): Please do problem 1.2.33 in 'Introduction to Real Analysis'.

Solution: We reduce the problem to the case in which the sets are disjoint. Consider the set

$$Z = \cup_{m,n=1}^{\infty} E_m \cap E_n$$

This is a countable union of sets of measure zero and by countable subadditivity $|Z| = 0$. Next, we remove Z from our considerations, i.e., we consider

$$G_k = E_k \setminus Z \ .$$

These sets are disjoint and $|G_k| = |E_k|$ Hence

$$| \cup_{k=1}^{\infty} E_k| = | \cup_{k=1}^{\infty} G_k \cup Z| = | \cup_{k=1}^{\infty} G_k| = \sum_{k=1}^{\infty} |G_k| = \sum_{k=1}^{\infty} |E_k|$$

Problem 2 (5 points): Please do problem 1.2.35 in 'Introduction to Real Analysis'.

The problem is the following:

Let $E \subset \mathbb{R}^m$ and $F \subset \mathbb{F}^n$. Assume that $P(x, y)$ is a statement that is either true or false for each pair $(x, y) \in E \times F$. Suppose that "for every $x \in E$, $P(x, y)$ is true for a.e. $y \in F$". Must it be true that " for a.e. $y \in F$, $P(x, y)$ is true for every $x \in E$"?

Solution: No, the second statement must not be necessarily true. Take the space $[0, 1] \times [0, 1]$ and consider the statement "$x \neq y$". For every fixed $x \in [0, 1]$, $x \neq y$ for almost every $y \in [0, 1]$. This is a true statement since the only point where the statement is false is the point $y = x$. Now for any y given, the statement $x \neq y$ is not true for every $x \in [0, 1]$, because $x = y$ violates this statement. Thus for any y the second statement is false. Here is another example I learned from C. Heil. Consider $E = F = \mathbb{R}$ and consider the statement $x - y \notin \mathbb{Q}$. For every $x \in \mathbb{R}$ we have that this is true for almost every $y \in \mathbb{R}$. However, for $y \in \mathbb{R}$ it is false that $x - y \notin \mathbb{Q}$ for all $x \in \mathbb{R}$, just pick $x = y + 2$, e.g..

Problem 3 (5 points): Please do problem 1.2.38 in 'Introduction to Real Analysis'.

Solution Recall Caratheodory's criterion, that E is measurable if and only if for ANY set $A \in \mathbb{R}^d$ we have that

$$|A|_e = |A \cap E|_e + |A \setminus E|_e \ .$$

Since E is measurable and $A \cap E = \emptyset$,

$$|E \cup A|_e = |(E \cup A) \setminus E|_e + |(E \cup A) \cap E|_e = |A|_e + |E| \ .$$

Another way of solving this is using that for the set $E \cup A$ there exists a G_δ set H with $E \cup A \subset H$ and $|E \cup A|_e = |H|$. Hence

$$|E \cup A|_e = |H| = |H \setminus E| + |E| \geq |A|_e + |E|$$

since $A \subset (H \setminus E)$. By subadditivity

$$|E \cup A|_e \leq |A|_e + |E| \; .$$

Problem 4 (7 points): Please do problem 1.2.43 in 'Introduction to Real Analysis'.

Solution: Assume that A, B are measurable then we have that $|E| = |A| + |B|$. For the converse we note that there exist G_δ set G and H with $A \subset G$ and $|A|_e = |G|$ and $B \subset H$ with $|B|_e = |H|$. We may assume that both G and H are subsets of E otherwise take the intersection with E.Note that E is a G_δ set up to a set of measure zero.

We have that

$$E \setminus G \subset B \subset H$$

and hence

$$|H \setminus B|_e \leq |H \setminus (E \setminus G)| = |H| - |E \setminus G)| = |H| - |E| + |G| = |A|_e + |B|_e - |E| = 0 \; .$$

Hence $H \setminus B$ is a set of measure zero and so B is measurable. The same argument applies to A.

Problem 5 (3 points): Please do problem 1.2.44 in 'Introduction to Real Analysis'.

Solution: Consider the set $E = \mathbb{Z}$ and consider the function $f : E \to \mathbb{R}$ given by $f(n) = |n|$. This function is continuous but the essential supremum is zero while the actual supremum is 'infinity'.