HOMEWORK 5, DUE MARCH 1

Problem 1 (5 points): Please solve Problem 3.1.11 in 'Introduction to real analysis'.

Solutions: a) We use Tchebyshev's inequality

$$|\{f > t\}| \leq \frac{1}{t} \int_E f$$

and hence

$$|\{f = \infty\}| = | \cap_{s>0} \{f > s\}| \leq \frac{1}{t} \int_E f$$

for any t and hence $| \cap_{s>0} \{f > s\}| = 0$.

b) Pick $\frac{1}{x^2}$ on the interval $[-1, 1]$.

Problem 2 (5 points): Please solve Problem 3.2.8 in 'Introduction to real analysis'.

Solution: Since the sequence $0 \leq f_n$ is pointwise monotone increasing it converges pointwise to some limit function $0 \leq f$ which might be intinite at certain points. By Fatou's lemma

$$\lim_{n\to\infty} \int_E f_n \geq \int_E f \ .$$

Because $f \geq f_n$ for all n we have that $\int_E f_n \leq \int_E f$ and thus,

$$\lim_{n\to\infty} \int_E f_n \leq \int_E f \ .$$

Problem 3 (5 points): Let $\{f_n\}_{n=1}^{\infty}$ is a sequence of non-negative measurable functions on a measurable set $E \subset \mathbb{R}^d$. Prove that

$$\int_E \sum_{j=1}^{\infty} f_n = \sum_{j=1}^{\infty} \int_E f_n.$$

Solution: Apply the monotone convergence theorem to the partial sums

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \ .$$

Problem 4 (5 points): Consider a non-negative integrable function $f : E \to \mathbb{R}_+$. where $E \subset \mathbb{R}^d$ is measurable. Show that

$$\int_E f = \int_0^\infty |\{x \in E : f(x) > t\}| dt$$

where the last integral is a Riemann integral.

Solution: The function $t \to |\{x \in E : f(x) > t\}|$ is monotone decreasing and hence Riemann integrable. Fix L large consider the sets $E_k = \{f > L\frac{k}{n}\}$. These sets define a sumple function given by

$$\phi_n = \frac{L}{n} \sum_{k=1}^{n} \chi_{E_k} \ .$$

Note that this simple function is not in the standard representation. But it is integrable and we get

$$\int_E \phi_n = \frac{L}{n} \sum_{k=1}^{n} |E_k| \ .$$

Moreover, we find that ϕ_n converges monotonically to f on the set where $f \le L$. Clearly $\int_E \phi_\le \int_E f$ and moreover $\int_E \phi_n \to \int_0^L |\{f > t\}|$ as $n \to \infty$. By monotone convergence

$$\int_{f \le L} f = \lim_{n \to \infty} \int_E \phi_n = \int_0^L |\{f > t\} dt \ .$$

Problem 5 (5 points): Use the result of Problem 4 to compute the Lebesgue integral

$$\int_{-1}^{1} |x|^{-1/2} dx \ .$$

Solution: We have that

$$|\{|x|^{-1/2} > t\}| = \frac{2}{t^2}$$

for $t > 1$ and equals 2 for $0 < t \le 1$. Hence,

$$\int_0^\infty |\{|x|^{-1/2} > t\}| dt = 2 + 2 \int_1^\infty \frac{1}{t^2} dt = 4 \ .$$