HOMEWORK 6, DUE WEDNESDAY MARCH 8

Problem 1 (5 points): Show that any function which is monotone on the interval $[a, b]$ is Riemann integrable on $[a, b]$.

Solution: Assume that f is monotone increasing. We have to present a sequence of partitions such that the lower sum and the upper sum get as close as we like. Pick a positive integer N and set

$$x_k = a + \frac{k}{N}(b-a) \ , k = 0, \dots, N \ .$$

These points define a partition Γ. The lower sum is given by

$$S_L = \sum_{j=1}^{N} f(x_{j-1})(x_j - x_{j-1}) = \sum_{j=1}^{N} f(x_{j-1}) \frac{b-a}{N}$$

and the upper sum by

$$S_U = \sum_{j=1}^{N} f(x_j)(x_j - x_{j-1}) = \sum_{j=1}^{N} f(x_j) \frac{b-a}{N} \ .$$

Hence

$$0 \le S_U - S_L = \sum_{j=1}^{N} f(x_j) \frac{b-a}{N} - \sum_{j=1}^{N} f(x_{j-1}) \frac{b-a}{N} = \frac{(f(b) - f(a))(b-a)}{N} \to 0, N \to \infty \ .$$

Problem 2 (7 points): Please do problem 3.5.13 in 'Introduction to real analysis'.

Solution: Consider the set $A = \{f > t\}$, where $t > 0$. Then, by assumption

$$0 = \int_A f \ge t |\{f > t\}| \ .$$

Hence the set where f is strictly positive has measure zero. Hence $f \le 0$, i.e., $f = -f^-$. Repeating the same argument for f^- yields the result. For the second part note that

$$\int_{|f|>t} |f| + \int_{|f| \le t} |f| = \int |f|$$

As $t \to \infty$ the second term tends to $\int |f|$ by monotone convergence. Hence the first term tends to zero and if we choose t large so that the first term is less than ε then the set $A = \{|f| \le t\}$ does the job.

Problem 3 (2 points): Please do problem 3.5.17 in 'Introduction to real analysis'.

Solution: Pick $E = \mathbb{R}$ and the sequence

$$f_n(x) = \frac{1}{n}\chi_{[-n,n]}(x) \ .$$

Problem 4 (8 points): Please do problem 3.5.24 in 'Introduction to real analysis'.

Solution: Here we imitate the proof of the dominated convergence theorem using Fatou's lemma. We know that

$$|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq \lim_{n\to\infty} g_n(x) = g(x) \ .$$

The function $g + g_n - |f - f_n|$ is non-negative and converges pointwise to $2g$. Hence, by Fatou's lemma

$$2\int g \leq \liminf_{n\to\infty} \int [g + g_n - |f - f_n|] = 2\int g - \limsup_{n\to\infty} \int |f - f_n|$$

which implies the result.

Problem 5 (3 points): Please do problem 3.5.26 in 'Introduction to real analysis'.

Solution: We may assume that f is real valued with $|f| \leq M$. Pick $\varepsilon > 0$. There exists a contnuous function g such that $\|f - g\| < \varepsilon$. By the Weierstrass theorem there exists a polynomial p such that $|g(x) - p(x)| < \varepsilon$ uniformly on $[0, 1]$. Hence

$$\int_0^1 f(x)^2 = \int_0^1 f(x)(f(x) - g(x)) + \int_0^1 f(x)(g(x) - p(x)) + \int_0^1 f(x)p(x) \ .$$

By assumption $\int_0^1 f(x)p(x) = \sum c_n \int f(x)x^n = 0$ and therefore

$$\int_0^1 f(x)^2 \leq M\int_0^1 |f(x) - g(x)| + M\int_0^1 |g(x) - p(x)| \leq 2M\varepsilon$$