HOMEWORK 7, DUE FRIDAY MARCH 17

Problem 1 (5 points): Please do problem 3.6.12 in 'Introduction to Real Analysis'.

Solution: For each x fixed the integral $\int_0^1 f(x,y)dy = 0$ and similarly for each y fixed the integral $\int_0^1 f(x,y)dx = 0$. However, the integral $\int_{Q_i} |f(x,y)|dxdy = 1$ and hence The total integral over the full square diverges. In particular $\int_Q f(x,y)^+ = \infty$ and $\int_Q f(x,y)^- = \infty$ and the integral $\int_Q f$ is not defined.

Problem 2 (6 points): Please do problem 3.6.21 in 'Introduction to Real Analysis'.

Solution: We just solve b) the other parts are proved by changing variables.

$$\begin{aligned} &[(f \star g) \star h](x) = \in_{\mathbb{R}^d} f \star g)(y)h(x-y)dy \\ = &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(z)g(y-z)dzh(x-y)dy \end{aligned}$$

Now using Fubini's theorem this equals

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(z)g(y-z)h(x-y)dydz \\ = &\int_{\mathbb{R}^d} f(z) \int_{\mathbb{R}^d} g(y-z)h(x-y)dydz \end{aligned}$$

and changing variables $y' = y - z$yields

$$\begin{aligned} &\int_{\mathbb{R}^d} f(z) \int_{\mathbb{R}^d} g(y')h(x-z-y')dy'dz \\ = &\int_{\mathbb{R}^d} f(z)(g \star h)(x-z)dz \\ = &[f \star (g \star h)](x) \ . \end{aligned} \tag{1}$$

Problem 3 (6 points): Please do problem 3.6.22 in 'Introduction to Real Analysis'.

Solution: a) Fix any $x \in \mathbb{R}^d$. Then $g(x-y)$ is, as a function of y in L^∞. Hence the function $f(y)g(x-y)$ is integrable for every $x \in \mathbb{R}^d$. Hence $f \star g(x)$ exists for *every* $x \in \mathbb{R}^d$.

b) We write

$$f \star g(x+h) - f \star g(x) = \int_{\mathbb{R}^d} f(y)g(x+h-y)dy - \int_{\mathbb{R}^d} f(y)g(x-y)dy$$

and changing variables $y' = x + h - y$ in the first and $y' = x - y$ in the second integral, this equals

$$\int_{\mathbb{R}^d} f(x+h-y')g(y')dy' - \int_{\mathbb{R}^d} f(x-y')g(y')dy' \ .$$

Hence

$$|f \star g(x+h) - f \star g(x)| \le \int_{\mathbb{R}^d} |f(x+h-y') - f(x-y')||g(y')|dy'$$

$$\le \|g\|_\infty \int_{\mathbb{R}^d} |f(x+h-y') - f(x-y')|dy' = \|g\|_\infty \int_{\mathbb{R}^d} |f(h-y) - f(y)|dy \to 0$$

as $h \to 0$.

c) Note that $|f \star g(x)| \le \int_{\mathbb{R}^d} |f(y)||g(x-y)|dy \le \|g\|_\infty \int_{\mathbb{R}^d} |f(y)|dy = \|g\|_\infty \|f\|_1$. Hence

$$\|f \star g\|_\infty \le \|g\|_\infty \|f\|_1 \ .$$

Problem 4 (8 points): Please do problem 3.6.23 in 'Introduction to Real Analysis'.

Solution: That $\chi_E \star \chi_{-E}(x)$ is continuous follows from the previous problem. Suppose that $\chi_E \star \chi_{-E}(x) \neq 0$. Then $x \in E - E$. For if $x \notin E - E$ then for any $y \in E$, $y - x$ is not in E and hence $\chi_E(y)\chi_{-E}(x-y) = 0$ and thus $\int \chi_E(y)\chi_{-E}(x-y)dy = 0$. As a consequence, we have to show that the set of all x with $\chi_E \star \chi_{-E}(x) \neq 0$ contains an interval. However, $\chi_E \star \chi_{-E}(0) = |E| > 0$ and since $\chi_E \star \chi_{-E}(x)$ is continuous, and the set contains an interval.