HOMEWORK 8, DUE FRIDAY MARCH 31

Problem 1 (5 points): Please do problem 4.1.6 in 'Introduction to real analysis'

Solution: Reflect the Cantor function about the y axis. The function has a derivative that vanishes almost everywhere. It grow from 0 to 1 and then back down to zero.

Problem 2 (5 points): Please do problem 4.2.10 in 'Introduction to real analysis'

Solution: a) Pick the partition $\Gamma = \{a, b\}$. Then $|f(b) - f(a)| = S_\Gamma \leq V[f; a, b]$.

b) Suppose that $\Gamma = \{a = x_0 < x_1 < \cdots < x_n = b\}$ now add a point y so that $\Gamma' = \{a = x_0 < x_1 < \cdots x_{i-1} < y < x_i < \cdots < x_n = b\}$. Then

$$|f(x_i) - f(x_{i-1})| \leq |f(x_i) - f(y)| + |f(y) - f(x_{i-1})|$$

and hence $S_\Gamma \leq S_{\Gamma'}$. The proof for an arbitrary refinement follows by induction.

c) For any partition Γ of $[c, d]$, $\Gamma' = \{a, b\} \cup \Gamma$ is a partition for $[a, b]$ and hence

$$S_\Gamma \leq S_{\Gamma'} \leq V[f; a, b] \ .$$

Hence $V[f; c, d] \leq V[f; a, b]$.

Problem 3 (5 points): Please do problem 4.2.16 in 'Introduction to real analysis'

Solution: If f_r, f_i are both in $BV[a, b]$ then since $|f(x) - f(y)| \leq |f_r(x) - f_r(y)| + |f_i(x) - f_i(y)|$ $f \in BV[a, b]$ as well. The converse follows since $|f_r(x) - f_r(y)| \leq |f(x) - f(y)|$ and $|f_i(x) - f_i(y)| \leq |f(x) - f(y)|$.

Problem 4 (5 points): Please do problem 4.2.20 in 'Introduction to real analysis'

Solution: The problem is that we only know that f is Lipschitz on A. For $E \subset A$, pick any $\varepsilon > 0$ and U open, so that $E \subset U$ and such that $|U| \leq |E|_e + \varepsilon$. The set U is a countable union of open intervals (a_n, b_n). The union of the sets $(a_n, b_n) \cap A$ contains E, but on these sets the function f is Lipschitz. For any $x, y \in (a_n, b_n) \cap A$ we have that

$$|f(x) - f(y)| \leq K(x - y| \leq K(b_n - a_n) \ .$$

Hence, $f((a_n, b_n) \cap A)$ is a subset of an interval of length at most $K(b_n - a_n)$ and therefore

$$|f(E)|_e \leq |f(U \cap A)|_e \leq \sum_n |f((a_n, b_n) \cap A)|_e \leq K \sum_n (b_n - a_n) = K|U| \leq K(|E|_e + \varepsilon)$$

and since ε is arbitrarily small the result follows.

Problem 5 (5 points): Please do problem 4.2.21 a) in 'Introduction to real analysis'

Solution: Suppose that $a \le b$. We want to show that f is not in $BV[-1,1]$. It suffices to show that the function restricted to $x \ge 0$ is not in $BV[0,1]$. Since $\sin \pi k = 0$ for all $k \in \mathbb{Z}$ we have that the function f vanishes at the values $x_k = \left(\frac{1}{\pi k}\right)^{1/b}$. Moreover, at the values $y_k = \left(\frac{1}{\pi(k+1/2)}\right)^{1/b}$ the sine function is ± 1. Hence

$$\sum_k |f(x_k) - f(y_k)| = \sum_k y_k^a = \sum_k \left(\frac{1}{\pi(k+1/2)}\right)^{a/b}$$

which, since $a/b \le 1$ diverges. For $a/b > 1$ let $\Gamma = \{0 = x_0 < x_1 \cdots < x_n = 1\}$ be any partition. For $x > 0$ the function f is continuously differentiable with derivative $ax^{a-1}\sin x^{-b} - bx^{a-b-1}\cos x^{-b}$ and hence

$$\sum_{j=1}^n |f(x_j) - f(x_{j-1}| = |f(x_1)| + \sum_{j=1}^n |\int_{x_{j-1}}^{x_j} f'(y)dy| \le |f(x_1)| + \sum_{j=1}^n \int_{x_{j-1}}^{x_j} |f'(y)|dy$$

Now $ax^{a-1}\sin x^{-b} - bx^{a-b-1}\cos x^{-b} \le ax^{a-1} + bx^{a-b-1}$ and hence

$$\sum_{j=1}^n \int_{x_{j-1}}^{x_j} |f'(y)|dy \le \int_{x_1}^1 [ax^{a-1} + bx^{a-b-1}]dx$$

and since $a > b$ we get for the integral

$$(1 - x_1^a) + \frac{b}{a-b}(1 - x_1^{a-b}) \le \frac{a}{a-b}\ .$$

The value of $|f(x_1)| \le 1$ and hence

$$S_\Gamma \le 1 + \frac{a}{a-b}$$

and $f \in BV[0,1]$.