HOMEWORK 9, DUE WEDNESDAY APRIL 12

Problem 5.1.7 (5 points): Pick $\varepsilon > 0$ and let $\sum(b_j - a_j) < \delta$ so that

$$\sum |f(b_j) - f(a_j)| < \varepsilon$$

Since the real part f_r satisfies $|f_r(b_0 - f_r(a_j)| \leq |f(b_j) - f(a_j)|$ it follows that

$$\sum |f_r(b_j) - f_r(a_j)| < \varepsilon \ .$$

The argument for the imaginary part f_i is similar.

Problem 5.1.10 (5 points): Assume that for any ε there exists $\delta > 0$ such that for *any finite* collection, of non-overlapping intervals $[b_j, a_j]$ with $\sum(b_j - a_j) < \delta$ it follows that $\sum |f(b_j) - f(a_j)| \leq \varepsilon$. Now pick a countable collection of non overlapping intervals with $\sum_1^\infty (b_j - a_j) < \delta$. Then we also have that $\sum_1^N (b_j - a_j) < \delta$ for any $N < \infty$. Hence we know that $c_N = \sum_1^N |f(b_j) - f(a_j)| \leq \varepsilon$ for any $N < \infty$. Since the sequence c_N is monotone increasing and $c_N \leq \varepsilon$ for all $N < \infty$ we also have that $\lim_{N\to\infty} c_N \leq \varepsilon$. Hence the function f is absolutely continuous. The converse of the statement is trivial.

Problem 5.3.5 a) (5 points): Let $g : [a, b] \to [c, d]$ be absolutely continuous and $f : [c, d] \to \mathbb{R}$ be Lipschitz with constant K. Let ε be given. We know that there exists $\delta > 0$ so that if $[a_j, b_j]$ is any collection of non-overlapping intervals in $[a, b]$ with $\sum(b_j - a_j) < \delta$, it follows that $\sum |g(b_j) - g(a_j)| \leq \varepsilon$. Hence

$$\sum |f(g(b_j)) - f(g(a_j))| \leq K \sum |g(b_j) - g(a_j)| \leq K\varepsilon$$

and hence $f \circ g$ is absolutely continuous.

Problem 5.3.8 (5 points): We have shown in class that if f is absolutely continuous and $[a, b]$ and f' which exists *a.e.* vanishes a.e., then f is constant everywhere. Now define $G(x) = \int_a^x g(y)dy$ and note that, by assumption $F' = g$ a.e.. Hence $F - G$, which is absolutely continuous, has a derivative that vanishes a.e., and hence $F - G$ is constant everywhere. Hence $F = G + c$, c some constant, and F is continuously differentiable.

Problem 5.3.11 (5 points): $|g(x)| \leq x^2$ and hence $g \in L^1[-1, 1]$. The function g is differentiable at every point, but the derivative for $x \neq 0$ is given by

$$2x \sin\left(\frac{1}{x^2}\right) - 2\frac{1}{x} \cos\left(\frac{1}{x^2}\right)$$

This function is not in L^1 and hence g is not absolutely continuous. One way to see that the function is not in L^1 is to compute with the substitution $s = 1/x^2$,

$$\int_{\varepsilon} \frac{1}{x} | \cos\left(\frac{1}{x^2}\right) | dx = \frac{1}{2} \int_{1}^{1/\varepsilon} | \cos(s) | \frac{1}{s} ds$$

and this last integral is easily seen to diverge as $\varepsilon \to 0$.