MIDTERM EXAM FOR MATH 6337, REAL ANALYSIS 1, FEBRUARY 15, 2017

Name:

Write legibly, write your arguments short and clearly! If I cannot read what you write or I cannot understand what you write I do not give credit. You have to convince me that your argument is right. It is not my job to show that your argument is wrong.

Problem 1 (5 points): Let $C \subset \mathbb{R}^d$ be compact and $f : C \to \mathbb{R}$ an upper semicontinuous function. Prove that f attains its maximum.

Solution: By definition of the infimum there exists a sequence $x_n \in C$ such that

$$\lim_{n\to\infty} f(x_n) = \inf_{y\in C} f(y) \ .$$

Since C is compact, there exists a subsequence x_{n_k} and $x \in C$ such that $\lim_{k\to\infty} x_{n_k} = x$. Because f is lower semicontinuous

$$\inf_{y\in C} f(y) = \lim_{k\to\infty} f(x_{n_k}) \geq f(x) \ ,$$

but $f(x) \geq \inf_{y\in C} f(y)$ and hence there must be equality. Thus, $f(x) = \inf_{y\in C} f(y)$.

Problem 2 (5 points): Let $E_1 \supset E_2 \supset E_3 \cdots$ be a countable collection of nested measurable sets in $\mathbb{R}^d$. Prove that

$$| \cap_{k=1}^\infty E_k| = \lim_{k\to\infty} |E_k| \ .$$

Solution: We set $B_j = E_j \setminus E_{j+1}$ and note that $E_k = (\cap_{i=1}^\infty E_i) \cup (\cup_{j=k+1}^\infty B_j)$. Since the union is disjoint

$$|E_k| = |(\cap_{i=1}^\infty E_i| + \sum_{j=k+1}^\infty |B_j|$$

and as $k \to \infty$, $\sum_{j=k+1}^\infty |B_j| \to 0$ and hence $|E_k| \to |(\cap_{i=1}^\infty E_i|$.

Problem 3 (5 points): Let A be a set in $\mathbb{R}^d$ with $|A|_e < \infty$. Suppose there exists an F_σ set $F \subset A$ such that $|F| = |A|_e$. Show that A is measurable.

Solution: We know that there exists a G_δ set H with $A \subset H$ and $|A|_e = |H|$. Note that H is measurable. Because $F \subset A$ and $|F| = |A|_e$ we have that $F \subset H$ and $|F| = |H|$. Hence $H = (H \setminus F) \cup F$, $|H| = |H \setminus F| + |F|$ and $|H \setminus F| = 0$. Since $A \setminus F \subset H \setminus F$, $A = F \cup (A \setminus F)$ where F is measurable and $A \setminus F$ has measure zero.

Problem 4 (5 points): Let $A, B \subset \mathbb{R}^d$ be measurable sets. Define the symmetric difference

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

Prove that $|A \Delta B| = 0$ if and only if

$$|A \cap B| = \frac{|A| + |B|}{2} \ .$$

Solution: the sets are measurable. We have that

$$|A \Delta B| = |A \setminus B| + |B \setminus A|$$

since the two sets are disjoint. We also have that $|A \setminus B| = |A| - |A \cap B|$ and $|B \setminus A| = |B| - |A \cap B|$. Hence,

$$|A \Delta B| = |A| + |B| - 2|A \cap B| \ .$$

From this formula the claim follows.

Problem 5 (5 points): Let $f : \mathbb{R}^d \to \mathbb{R}$ be a lower semicontinuous function. Show that f is measurable.

Solution: A lower semicontinuous function has open level sets and hence it is measurable.

Problem 6 (3 points each): True or false. You do not have to prove anything.

a) For any subset $A \subset \mathbb{R}^d$ there exists a G_δ set H such that $A \subset H$ and $|H \setminus A|_e = 0$. FALSE

b) The pointwise limit of a sequence of measurable functions is measurable. TRUE (Note: I should have added that the functions are finite a.e. in the statement of the problem).

c) A property holds almost everywhere if the set where the property fails has measure zero. TRUE

d) If $f_n, f : E \to \mathbb{R}$ are non-negative functions and f_n converges pointwise to f then $\int_E f_n \to \int_E f$ as $n \to \infty$. FALSE

e) Every sequence that converges in measure converges pointwise a.e. FALSE