PRACTICE FINAL EXAM FOR MATH 6337, REAL ANALYSIS 1, APRIL 27, 2017

Name:

Write legibly and write your arguments short and clearly! If I cannot read what you write or I cannot understand what you write I do not give credit. You have to convince me that your argument is right. It is not my job to show that your argument is wrong.

Problem 1 (5 points): Assume that $1 \le p < q \le \infty$ and that $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Show that $f \in L^r(\mathbb{R}^d)$ for all $p \le r \le q$.

Solution: Since $p \le r \le q$ we mat write

$$r = tp + (1-t)q$$

or by setting $t = \frac{1}{s}$ we have that $1 - t = \frac{1}{s'}$ where s' is the index dual to s. Now

$$\int |f|^r = \int (|f|^p)^{\frac{1}{s}} (|f|^q)^{\frac{1}{s'}} \le (\int |f|^p)^{\frac{1}{s}} (\int |f|^q)^{\frac{1}{s'}}$$

Problem 2 (5 points): For $x > 0$ consider the function $s(x) = x \log x$ and set $s(0) = 0$. Show that on $(0, \infty)$ this function is convex and show that for any function $f \ge 0$ measurable on a set E with $|E| < \infty$ with $\frac{1}{|E|} \int_E f = 1$,

$$\frac{1}{|E|} \int_E s(f(x)) dx \ge 0$$

Solution: The function $x \log x$ has the derivative $\log x + 1$ and its second derivative is $1/x > 0$. Hence the function is convex. Now Jensen's inequality yields

$$\frac{1}{|E|} \int_E s(f(x)) dx \ge s(\frac{1}{|E|} \int_E f(x) dx) = \left(\frac{1}{|E|} \int_E f(x) dx \right) \log \left(\frac{1}{|E|} \int_E f(x) dx \right) = 0$$

because $\frac{1}{|E|} \int_E f(x) dx = 1$.

Problem 3 (5 points): You have shown in one of the exercises that for a non-negative integrable function on $\mathbb{R}^d$ one has the formula

$$\int_0^\infty |\{f > a\}| da \ .$$

Find an analogous formula for $\|f\|_p$, the $L^p(\mathbb{R}^d)$ norm of a function. Use this to show that when f, g are two non-negative functions with $|\{f > a\}| = |\{g > a\}|$ then their p norms are the same.

Solution: We start with

$$\int |f|^p = \int_0^\infty |\{|f|^p > a\}| da$$

now set $a = b^p$ and note that $\{|f|^p > a\} = \{|f| > b\}$ so that

$$\int_0^\infty |\{|f|^p > a\}| da = p \int_0^\infty |\{|f| > b\}| b^{p-1} db$$

This means that the L^p norm of a function depends only on the measures of the level sets. Hence if $|\{f > a\}| = |\{g > a\}|$ then the p-norms of g and f are the same.

Problem 4 (5 points): Let $f \in L^p(\mathbb{R}^d)$ for all p sufficiently large. Show that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty \ .$$

Solution: Consider the set

$$E_\varepsilon = \{x \in \mathbb{R}^d : |f(x)| \geq \|f\|_\infty - \varepsilon\}$$

We note that for some fixed q large

$$(\|f\|_\infty - \varepsilon)|E_\varepsilon| \leq \int_{E_\varepsilon} |f|^q \leq \int |f|^q < \infty \ .$$

and hence $|E_\varepsilon| \leq C$. Moreover, $|E_\varepsilon| > 0$, because otherwise the essential supremum of $|f|$ would be smaller than $\|f\|_\infty$. Further

$$\liminf_{p \to \infty} (\int |f|^p)^{1/p} \geq \lim_{p \to \infty} (\|f\|_\infty - \varepsilon)|E_\varepsilon|^{1/p} = \|f\|_\infty - \varepsilon$$

Thus,

$$\liminf_{p \to \infty} (\int |f|^p)^{1/p} \geq \|f\|_\infty \ .$$

Next, suppose that $\limsup_{p \to \infty} (\int |f|^p)^{1/p} > \|f\|_\infty$. Fix some r large with $\|f\|_r < \infty$ and pick $0 < \varepsilon < \|f\|_\infty$. Then for $p > r$

$$\|f\|_p = \left(\int_{|f| > \varepsilon} |f|^p + \int_{|f| \leq \varepsilon} |f|^p \right)^{1/p} \leq \left(|\{|f| > \varepsilon\}| \|f\|_\infty^p + \varepsilon^{p-r} \int |f|^r \right)^{1/p}$$

$$= \|f\|_\infty \left(|\{|f| > \varepsilon\}| + \frac{\varepsilon^{p-r} \int |f|^r}{\|f\|_\infty^p} \right)^{1/p} \leq \|f\|_\infty \left(|\{|f| > \varepsilon\}| + \varepsilon^{-r} \int |f|^r \right)^{1/p}$$

Hence

$$\limsup_{p \to \infty} \|f\|_p \leq \|f\|_\infty \limsup_{p \to \infty} \left(|\{|f| > \varepsilon\}| + \varepsilon^{-r} \int |f|^r \right)^{1/p} = \|f\|_\infty \ .$$

Problem 5 (5 points): Let f, g be two measurable functions on $\mathbb{R}^d$. Prove that the set $\{x : f(x) > g(x)\}$ is measurable.

Solution: We write

$$\{x : f(x) > g(x)\} = \cup_{r\in\mathbb{Q}} \left[\{f(x) > r\} \cap \{g(x) < r\}\right]$$

and since $\mathbb{Q}$ is countable and f as well as g are measurable the set $\{x : f(x) > g(x)\}$ is measurable too.

Problem 6 (5 points): Let $E \subset \mathbb{R}^d$ be a measurable set with $|E| < \infty$ and let f_j be a sequence of complex valued functions such that $f_j \to f$ pointwise almost everywhere. Assume that $\int_E |f_j|^2 \leq 1$ and $\int_E |f|^2 < \infty$. Show that $\int_E |f - f_j|^p \to 0$ for all $1 \leq p < 2$.

Solution: Pick any $\varepsilon > 0$. By Egorov's theorem there exists a set $A \subset E$ such that $|E \setminus A| < \varepsilon$ and $f_j \to f$ uniformly on A. Thus,

$$\limsup_{j\to\infty} \int_E |f_j - f|^p \leq \limsup_{j\to\infty} \int_A |f_j - f|^p + \limsup_{j\to\infty} \int_{E\setminus A} |f_j - f|^p$$

However,

$$\int_{E\setminus A} |f_j - f|^p = \int_{E\setminus A} |f_j - f|^p \cdot 1 \leq [\int_{E\setminus A} |f_j - f|^2]^{1/2} [\int_{E\setminus A} 1]^{1/r} = [\int_{E\setminus A} |f_j - f|^2]^{1/2} \varepsilon^{1/r}$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{2} > 0$. By assumption

$$\|f_j - f\|_2 \leq \|f\|_2 + \|f_j\|_2 \leq \text{Constant}$$

where the constant is independent of j. Moreover

$$\limsup_{j\to\infty} \int_A |f_j - f|^p = 0$$

because of the uniform convergence. Hence,

$$\limsup_{j\to\infty} \int_E |f_j - f|^p \leq \text{Constant}\varepsilon^{1/r}$$

and since ε is arbitrary the result follows.

Problem 7 (5 points): Let ϕ be a non-negative function with compact support on $\mathbb{R}^d$ and assume that $\int \phi(y)dy = 1$. Let $\varepsilon > 0$ and consider $\phi_\varepsilon(x)\varepsilon^{-d}\phi(\frac{x}{\varepsilon})$. For $f \in L^1(\mathbb{R}^d)$ show that

$$\|\phi_\varepsilon \star f - f\|_1 \to 0$$

as $\varepsilon \to 0$.

Solution: We write

$$\phi_\varepsilon \star f - f = \int \phi_\varepsilon(y)[f(x - y) - f(x)]dy$$

and changing variables this equals

$$\int \phi(y)[f(x-\varepsilon y) - f(x)]dy \ .$$

Integrating with respect to x

$$\int |\int \phi(y)[f(x-\varepsilon y) - f(x)]dy| dx$$

and using Fubini's theorem, we find the bound

$$\int \phi(y) \int |f(x-\varepsilon y) - f(x)| dx dy \ .$$

The integrand with respect to the variable y is bounded by $2\phi(y)\|f\|_1$ and since ϕ has compact support we find that

$$\lim_{\varepsilon \to 0} \int \phi(y) \int |f(x-\varepsilon y) - f(x)| dx dy = \int \phi(y) \lim_{\varepsilon \to 0} \int |f(x-\varepsilon y) - f(x)| dx dy = 0 \ .$$

Problem 8 (5 points): Let ϕ be a function in $C^1(\mathbb{R}^d)$ with compact support. Show that for any $f \in L^1(\mathbb{R}^d)$ the function

$$\phi \star f(x)$$

is differentiable.

Solution: First we note that

$$\lim_{h \to 0} \frac{\phi(x+h) - \phi(x) - \nabla\phi(x) \cdot h}{|h|} = 0$$

since ϕ is differentiable. Next we write

$$\phi(x+h) - \phi(x) = \int_0^1 \nabla\phi(x+th) \cdot h dt$$

and find that

$$|\frac{\phi(x+h) - \phi(x) - \nabla\phi(x) \cdot h}{|h|}| \le C$$

for all h and all x where C is some constant. This uses the assumption that ϕ has compact support. Now,

$$\frac{1}{|h|}\left[\phi \star f(x+h) - \phi \star f(x) - \int \nabla\phi(x-y) \cdot h f(y) dy\right]$$
$$= \int \frac{1}{|h|}\left[\phi(x+h-y) - \phi(x-y) - \nabla\phi(x-y) \cdot h\right] f(y) dy$$

The integrand is bounded by a constant times $|f(y)|$ and as $h \to 0$ it tends to zero. Hence by dominated convergence we have that

$$\lim_{h \to 0} \frac{1}{|h|}\left[\phi \star f(x+h) - \phi \star f(x) - \int \nabla\phi(x-y) \cdot h f(y) dy\right] = 0$$

as claimed.

Problem 9 (5 points): Deduce the monotone convergence theorem from Fatou's lemma.

Solution: Let f_n be a monotone increasing sequence of functions. This sequence has a limit which we call f. The limit might be $+\infty$ on some set. Then

$$\lim_{n\to\infty} \int f_n = \liminf_{n\to\infty} \int f_n \geq \int \liminf_{n\to\infty} f_n = \int f$$

and hence if the left side is finite then so is the right and f is finite almost everywhere. On the other hand since $f_n \leq f$ we have that

$$\int f_n \leq \int f$$

and therefore

$$\lim_{n\to\infty} \int f_n \leq \int f \ .$$

Problem 10 (5 points): Let $E_k \subset \mathbb{R}^d$ be a sequence of measurable sets with $\sum_{k=1}^{\infty} |E_k| < \infty$. Then almost all $x \in \mathbb{R}^d$ are in at most finitely many of the sets E_k.

Solution: Consider the function

$$f(x) = \sum_{n=1}^{\infty} \chi_{E_n}(x)$$

where $\chi_{E_n}(x)$ is the characteristic function of the set E_n. Using the monotone convergence theorem we know that

$$\int f = \lim_{N\to\infty} \int \sum_{n=1}^{N} \chi_{E_n}(x) < \infty \ .$$

This means that the function f is finite almost everywhere. If x is a point that is in infintely many of the E_ns then f must diverge. But it diverges only on a set of measure zero and hence the set of x that belong to infinitely many of the E_ns is a set of measure zero.