Problem 1 (5 points): Let $C \subset \mathbb{R}^d$ be compact and $f : C \to \mathbb{R}$ a lower semicontinuous function. Prove that f attains its minimum.

Solution: Let $x_n \in C$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} f(x_n) = \inf_{x\in C} f(x) \ .$$

Since C is compact, it follows that there exists a sub-sequence, which we again denote by x_n and which converges to some point $x \in C$. Since f is lower semicontinuous,

$$\lim_{n\to\infty} f(x_n) \geq f(x)$$

and hence

$$\inf_{x\in C} f(x) \geq f(x)$$

and hence $f(x) = \inf_{x\in C} f(x)$.

Problem 2 (8 points): recall that a function $f : [a, b] \to \mathbb{R}$ is monotone increasing if $f(x) \leq f(y)$ whenever $x, y \in [a, b]$ and $x \leq y$. The definition of monotone decreasing is similar. Prove that any monotone function is Lebesgue measurable.

Solution: Consider the level set

$$\{a \leq x \leq b : f(x) \geq t\}$$

This set is either the empty set, the whole interval, an interval of the form $[c, b]$ or $(c, b]$. This follows from the monotonicity of f. All these sets are measurable.

Problem 3 (9 points): Let E be a measurable set in $\mathbb{R}^d$. Show that for any $\varepsilon > 0$ there exists an open set U and a closed set F such that $F \subset E \subset U$ and $|U \setminus F| < \varepsilon$.

Solution: For $\varepsilon > 0$ given there exists an open set U with $E \subset U$ and $|U \setminus E| < \varepsilon/2$. The set E^c is also measurable and hence there exists an open set U' with $E^c \subset U'$ such that $U' \setminus E^C| < \varepsilon/2$. The set $F = U'^c$ is closed with $F \subset E$. Since

$$U' \setminus E^c = E \setminus U'^c = E \setminus F$$

we have that

$$|E \setminus F| < \varepsilon/2 \ .$$

Finally

$$|U \setminus F| = |(U \setminus E) \cup (E \setminus F)| \leq |(U \setminus E)| + |(E \setminus F)| < \varepsilon \ .$$

Problem 4 (9 points): Let $E_1 \subset E_2 \subset \cdots$ be a sequence of nested measurable sets in $\mathbb{R}^d$. Prove that

$$| \cup_{k=1}^\infty E_k| = \lim_{n\to\infty} |E_n|$$

Solution: Define the sets $B_1 = E_1$ and for $j \geq 2$, $B_j = E_j \setminus E_{j-1}$ and note that these sets are disjoint and for all m

$$E_m \cup_{k=1}^m E_k = \cup_{j=1}^m B_j \ .$$

Hence,

$$| \cup_{k=1}^\infty E_k| = | \cup_{k=1}^\infty B_k| = \sum_{k=1}^\infty |B_k| = \lim_{n\to\infty} \sum_{k=1}^n |B_k| = \lim_{n\to\infty} |E_n| \ .$$

Problem 5 (9 points): In Egorov's theorem we had to assume that $|E| < \infty$. Give an example of a sequence of functions on the whole real line which converges but where Egorov's theorem fails.

Solution: Take $E = \mathbb{R}$ and consider the sequence $f_n(x) = \chi_{[-n,n]}(x)$. This sequence converges pointwise to 1 but not uniform on any unbounded set.