Homework 1, due Wednesday February 1

I: A linear operator $A : D(A) \to \mathcal{H}$ is closed if and only if the domain $D(A)$ endowed with the norm $\|f\|_A := \sqrt{\|f\|^2 + \|Af\|^2}$ is a Banach space, i.e., a linear, normed, complete space.

II: (Reed-Simon Vol. I) Let ϕ be a bounded measurable function on the real line and assume that ϕ is not in $L^2(\mathbb{R})$. Fix a function $\psi \in L^2(\mathbb{R})$ and consider the operator

$$Af = (\phi, f)\psi$$

on the domain

$$D(A) = \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |f(x)\phi(x)| dx < \infty\} \ .$$

It is clear that all bounded functions of compact support are in $D(A)$ and hence A is densely defined. Compute the adjoint A^*.

III: (Reed-Simon Vol. I) Let $\mathcal{H}$ be a Hilbert space and e_n and orthonormal basis in $\mathcal{H}$ and denote by e_∞ the vector

$$\sum_{n=1}^{\infty} \frac{1}{n} e_n \ .$$

Consider the domain D consisting of *finite* linear combinations of the form

$$a e_\infty + \sum c_i e_i \ ,$$

and on D consider the linear operator

$$A(a e_\infty + \sum c_i e_i) = a e_\infty \ .$$

Show that A is not closable.

IV: Let A be a densely defined operator on a Hilbert space $\mathcal{H}$. Show that

$$Ran(A)^\perp = Ker(A^*) \ .$$

Is it true that

$$Ker(A^*)^\perp = Ran(A) \ ?$$