Homework 1, due Wednesday February 1

I: A linear operator $A : D(A) \to \mathcal{H}$ is closed if and only if the domain $D(A)$ endowed with the norm $\|f\|_A := \sqrt{\|f\|^2 + \|Af\|^2}$ is a Banach space, i.e., a linear, normed, complete space.

Solution: Assume that A is closed. Let f_n be a Cauchy sequence, i.e., $\|f_n - f_m\|_A$ tends to zero as $n, m \to \infty$. Since $\mathcal{H}$ is complete, $f_n \to f$ and $Af_n \to g$ in $\mathcal{H}$. Since A is closed, $f \in D(A)$ and $Af = g$. Thus, $D(A)$ is a Banach space. Conversely, let $f_n \in D(A)$ with $f_n \to f$ and $Af_n \to g$. This means that f_n is a Cauchy sequence, i.e., $\|f_n - f_m\|_A$ tends to zero as $n, m \to \infty$. Since $D(A)$ is a Banach space we have that $f \in D(A)$ and $Af = g$ and A is closed.

II: (Reed-Simon Vol. I) Let ϕ be a bounded measurable function on the real line and assume that ϕ is not in $L^2(\mathbb{R})$. Fix a function $\psi \in L^2(\mathbb{R})$ and consider the operator

$$Af = (\phi, f)\psi$$

on the domain

$$D(A) = \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |f(x)\phi(x)| dx < \infty\} .$$

It is clear that all bounded functions of compact support are in $D(A)$ and hence A is densely defined. Compute the adjoint A^*.

Solution: Let $G \in D(A^*)$, i.e., $f \to (g, Af)$ extends to a bounded linear functional on all of $L^2(\mathbb{R})$. Thus,

$$(A^*g, f) = (\phi, f)(g, \psi) .$$

Thus, one might be tempted to write $A^*g = (\psi, g)\phi$, however, $\phi \notin L^2(\mathbb{R})$. Thus, the domain of A^* consists of all functions perpendicular to ψ and the operator A^* is the zero operator.

III: (Reed-Simon Vol. I) Let $\mathcal{H}$ be a Hilbert space and e_n and orthonormal basis in $\mathcal{H}$ and denote by e_∞ the vector

$$\sum_{n=1}^{\infty} \frac{1}{n} e_n .$$

Consider the domain D consisting of *finite* linear combinations of the form

$$a e_\infty + \sum c_i e_i ,$$

and on D consider the linear operator

$$A(a e_\infty + \sum c_i e_i) = a e_\infty .$$

Show that A is not closable.

Solution: Take the sequence

$$-a \sum_{n=1}^{N} \frac{1}{n} e_n$$

so that

$$f_N := a e_\infty - a \sum_{n=1}^{N} \frac{1}{n} e_n$$

is in D. The sequence converges to zero in $\mathcal{H}$ and the sequence

$$Af_n = ae_\infty$$

obviously converges. But this limit is not zero.

IV: Let A be a densely defined operator on a Hilbert space $\mathcal{H}$. Show that

$$Ran(A)^\perp = Ker(A^*) \ .$$

Is it true that

$$Ker(A^*)^\perp = Ran(A) \ ?$$

Solution: $f \in \mathrm{Ran}(A)^\perp$ means that $(f, Ag) = 0$ for all $g \in D(A)$. Hence, since A is densely defined, $f \in D(A^*)$ and $A^* f = 0$. Thus $\mathrm{Ran}(A)^\perp \subset \mathrm{Ker}(A^*)$. By the same formula, if $f \in \mathrm{Ker}(A^*)$ then $(f, Ag) = 0$ for all $G \in D(A)$ and hence $\mathrm{Ran}(A)^\perp = \mathrm{Ker}(A^*)$.

It is in general not true that $\mathrm{Ran}(A) = \mathrm{Ker}(A^*)^\perp$, because $\mathrm{Ran}(A)$ is in general not closed. As an example, take the operator on $L^2[0,1]$, $Af(x) = xf(x)$. The sequence of characteristic functions

$$\chi_{[1/n,1]}(x)$$

is in $\mathrm{Ran}(A)$ since they are the image of $\frac{1}{x}\chi_{[1/n,1]}(x) \in L^2[0,1]$. Thus sequence tends to the function 1 and there is no function f in $L^2[0,1]$ with $xf(x) = 1$.