Homework 2, due Wednesday March 1

I: Prove that if a closed symmetric operator A has a real number in its resolvent set, then it is self adjoint.

Solution: Let $\lambda \in \mathbb{R}$ be in the resolvent set. Then $(A - \lambda I)^{-1}$ exist as a bounded operator on all of $\mathcal{H}$. In particular this means that $\mathrm{Ran}(A - \lambda I) = \mathcal{H}$ and since $B = A - \lambda I$ is symmetric, it is self adjoint.

To see this, take $f \in D(B^*)$ and set $g = B^* f$. This is defined since $D(B) \subset D(B^*)$. Since B is onto, we may write $g = Bh$ for $h \in D(B)$. Thus, for any $u \in D(B)$ we have that

$$(g,u) = (B^* f, u) = (f, Bu) \ , \ (g,u) = (Bh, u) = (h, Bu)$$

and hence we have that $(f - h, Bu) = 0$ for all $u \in D(B)$. Since B is onto, $f = h$ and $f \in D(B)$.

II: Let A be a self adjoint operator and B be a symmetric operator which is A bounded with bound a. Show that

$$\limsup_{\mu \to \infty} \|B(A + i\mu I)^{-1}\| \le a \ .$$

Solution: We have

$$\|B(A + i\mu I)^{-1} f\| \le a\|A(A + i\mu I)^{-1} f\| + b\|(A + i\mu I)^{-1}\|$$

for all $f \in \mathcal{H}$. Further

$$\|f\|^2 = \|(A + i\mu I)(A + i\mu I)^{-1} f\|^2 \ge \|A(A + i\mu I)^{-1} f\|^2 + |\mu|^2 \|(A + i\mu I)^{-1} f\|^2$$

we find that

$$\|A(A + i\mu I)^{-1} f\| \le \|f\|$$

and

$$\|(A + i\mu I)^{-1} f\| \le \frac{\|f\|}{|\mu|} \ .$$

Hence

$$\|B(A + i\mu I)^{-1} f\| \le a\|f\| + \frac{b}{|\mu|}\|f\|$$

from which the result follows.

III: With the abbreviation $p = \frac{1}{i}\frac{d}{dx}$ consider the operator $A = px^5 + x^5 p$ on the domain $D(A) = C_c^\infty(\mathbb{R})$. Prove that A does not have any self adjoint extensions.

Solution: We have to solve the equation $(A^* \pm iI)f = 0$. We start by solving the equation

$$(x^5 f)' + x^5 f' = \mp f$$

which leads to

$$f = C|x|^{-5/2} e^{\pm \frac{1}{8|x|^2}} \ .$$

If we set $f(0) = 0$ we see that for the minus sign the function is in $L^2(\mathbb{R})$ but not for the plus sign. This suggests that the deficiency indices are $(1, 0)$ and hence there is no self adjoint extension.

To prove this is a bit tricky and not required for solving the homework problem. We give only a sketch of how to do that. The function f is $C^\infty(\mathbb{R})$ and if $g \in C_c^\infty(\mathbb{R})$ so is fg. For $h \in D(A^*)$ we consider the equation

$$(h, (A \pm iI)(fg)) = 0 \ .$$

First we note that

$$A(fg) = -igf + f\frac{1}{i}2x^5 g' \ .$$

All the computation are justified since the functions are smooth. Hence we get

$$(h, -ifg + f\frac{1}{i}2x^5 g' \pm iIfg) = 0$$

for all $g \in C_c^\infty(\mathbb{R})$. Chooing the $+$ sign we get that

$$(h, f\frac{1}{i}2x^5 g') = 0$$

which means that

$$\int_{\mathbb{R}} x^5 \overline{h(x)f(x)} g'(x) dx = 0$$

for all $g \in C_c^\infty(\mathbb{R})$. It is not difficult although one to think a little bit, that this implies that $x^5\overline{h(x)f(x)}$ is constant. Of course this constant should not be zero. Hence

$$h = C\frac{1}{x^5 f} \notin L^2(\mathbb{R})$$

and hence there is no solution in $L^2(\mathbb{R})$ for the equation $(A^* - iI)h = 0$. Now we choose the minus sign and get

$$\int_{\mathbb{R}} \overline{h(x)f(x)}[g + x^5 g'](x) dx = 0$$

for all $g \in C_c^\infty(\mathbb{R})$. This we can write

$$\int_{\mathbb{R}} \overline{h(x)f(x)} x^5 e^{\frac{1}{4x^4}} [e^{-\frac{1}{4x^4}} g]'(x) dx = 0 \ .$$

Once more the function $e^{-\frac{1}{4x^4}} g \in C_c^\infty(\mathbb{R})$ and since g is arbitrary we find again that

$$\overline{h(x)f(x)} x^5 e^{\frac{1}{4x^4}} = C$$

a constant. Solving for h yields

$$h(x) = C\frac{1}{x^5 f(x)} e^{-\frac{1}{4x^4}}$$

which is in $L^2(\mathbb{R})$.

IV: Let A be a symmetric operator on a Hilbert space. Prove that the following statements are equivalent:

a) A is essentially self adjoint.

b) $\mathrm{Ker}(A^* \pm iI) = \{0\}$

c) $\mathrm{Ran}(A \pm iI)$ is dense.

Solution: Assume that A is essentially self adjoint, i.e., $\overline{A}$ is self adjoint. Since $A^* = \overline{A}^*$ we have that $\mathrm{Ker}(A^* \pm iI) = \mathrm{Ker}(\overline{A} \pm iI) = \{0\}$. Since $\overline{\mathrm{Ran}(A \pm iI)} \oplus \mathrm{Ker}(A^* \pm iI) = \mathcal{H}$ it follows that $\mathrm{Ran}(A \pm iI)$ is dense if and only if $\mathrm{Ker}(A^* \pm iI) = \{0\}$. It remains to show that if $\overline{\mathrm{Ran}(A \pm iI)} = \mathcal{H}$ then $\overline{A}$ is self adjoint. This follows from $\overline{\mathrm{Ran}(A \pm iI)} = \mathrm{Ran}(\overline{A} \pm iI)$. To see this let $f \in \mathrm{Ran}(\overline{A} + iI)$. There exists $g \in D(\overline{A})$ such that $f = (\overline{A} + iI)g$. There exists a sequence $g_n \in D(A)$ such that $g_n \to g$ and $(A + iI)g_n \to f$. Hence $f \in \overline{\mathrm{Ran}(A + iI)}$. Since $\overline{\mathrm{Ran}(A \pm iI)} \subset \mathrm{Ran}(\overline{A} \pm iI)$ the result follows.

V (Reed-Simon Vol. II): Let A be a closed symmetric operator which has a self adjoint extension. Is it possible that A has a closed symmetric extension B that has no self adjoint extension? Explain.

Solution: This is best discussed on the level of the Cayley transform. Let V_A be the Cayley transform of A and denote by C a self adjoint extension of A. Then $V_A : F \to G$ is an isometry, where F, G are closed subspaces of $\mathcal{H}$ and V_A is onto G. Since C is self adjoint there exist $F_\pm$ such that $F \oplus F_+ = G \oplus F_- = \mathcal{H}$ and $\mathrm{dim} F_+ = \mathrm{dim} F_-$. The Cayley transform of B, V_B, is an isometry from the closed subspace P onto the closed subspace Q. We have that $P = F \oplus P_+$ and $Q = G \oplus P_-$ for some subspace $P_\pm$. Since V_B is an isometric extension of V_A we must have that $\mathrm{dim} P_+ = \mathrm{dim} P_-$. Since $P_\pm$ are subspaces of $F_\pm$ this would imply that $\mathrm{dim} F_+ - \mathrm{dim} P_+ = \mathrm{dim} F_- - \mathrm{dim} P_-$ and B would have self adjoint extensions, which contradicts our assumption.