Homework 3, due Wednesday March 29

I: Let $\mathcal{A}$ be a commutative Banach Algebra and $\mathcal{I}$ a closed ideal. Recall that any element in the factor algebra $\mathcal{A}/\mathcal{I}$ is given as an equivalence class $[x]$ where two elements $x, y \in \mathcal{A}$ are equivalent if $x - y \in \mathcal{I}$. Also recall that

$$\|[x]\| = \inf_{z \in \mathcal{I}} \|x - z\| \ .$$

The multiplication of equivalence classes are defined by $[x][y] = [xy]$. Show that $\mathcal{A}/\mathcal{I}$ is a Banach Algebra.

Solution: a) If x, y are elements in $\mathcal{A}$ we have that

$$[x] + [y] = x + \mathcal{I} + y + \mathcal{I} = [x + y] \ , [\lambda x] = \lambda x + \mathcal{I} = \lambda(x + \mathcal{I}) = \lambda[x] \ .$$

Likewise

$$[x][y] = (x + \mathcal{I})(y + \mathcal{I}) = xy + x\mathcal{I} + \mathcal{I}y + \mathcal{I} = [xy]$$

since $\mathcal{I}$ is an ideal. Hence $\mathcal{A}/\mathcal{I}$ is an algebra.

b) We have that $\|[x]\| = \inf_{z\in\mathcal{I}} \|x - z\| \geq 0$. Further $\|[0]\| = \inf_{z\in\mathcal{I}} \|0 - z\| = 0$ and if $\|[x]\| = 0$, there exists a sequence of elements $z_j \in \mathcal{I}$ so that $\|x - z_j\| \to 0$ as $j \to \infty$. This means that $z_j \to x$ and since $\mathcal{I}$ is closed, $x \in \mathcal{I}$ and hence $[x] = [0]$. If $\lambda \in \mathbb{C}$ we have that

$$\|\lambda[x]\| = \inf_{z\in\mathcal{I}} \|\lambda x - z\| = |\lambda| \inf_{z\in\mathcal{I}} \|\lambda x - \frac{z}{\lambda}\| = |\lambda|\|[x]\| \ .$$

If $\lambda = 0$ then $\|\lambda[x]\| = 0 = |\lambda|\|[x]\|$. It remains to show the triangle inequality $\|[x] + [y]\| = \inf_{z\in\mathcal{I}} \|x + y - z\| = \inf_{z,w\in\mathcal{I}} \|x - z + y - w\| \leq \inf_{z\in\mathcal{I}} \|x - z\| + \inf_{w\in\mathcal{I}} \|y - w\| = \|[x]\| + \|[y]\|$.

c) We have

$$\|[x][y]\| = \inf_{z\in\mathcal{I}} \|xy - z\| \ .$$

Pick $u, v \in \mathcal{I}$ and set $z = xu + vy - vu \in \mathcal{I}$. This element in $\mathcal{I}$ is of a special form and hence

$$\inf_{z\in\mathcal{I}} \|xy - z\| \leq \inf_{u,v\in\mathcal{I}} \|(x - v)(y - u)\| \leq \inf_{u,v\in\mathcal{I}} \|(x - v)\|\|(y - u)\| = \|[x]\|\|[y]\| \ .$$

d) We have to show that the quotient algebra is complete. Let $[x_n]$ be a Cauchy sequence. For any $\varepsilon > 0$ there exists N so that $\|[x_n] - [x_m]\| < \varepsilon$ for $n, m > N$. There exists a subsequence, again denoted by $[x_n]$ such that $\|[x_n] - [x_{n+1}]\| \leq 2^{-n}$. Now $\|[x_n] - [x_{n+1}]\| = \inf_{z\in\mathcal{I}} \|x_n - x_{n+1} - z\|$ and hence there are representatives in the class, which we call again x_n such that

$$\|x_n - x_{n+1}\| \leq 2^{-n+1} \ .$$

Thus, x_n is a Cauchy sequence in $\mathcal{A}$ and since $\mathcal{A}$ is complete there exists $x \in \mathcal{A}$ with $\|x_n - x\| \to 0$ as $n \to \infty$. Hence,

$$\|[x_n] - [x]\| = \inf_{z\in\mathcal{I}} \|x_n - x - z\| \leq \|x_n - x\| \to 0$$

as $n \to \infty$. It is then easy to see that the whole Cauchy sequence converges to $[x]$.

II: Let $\mathcal{A}$ be a commutative Banach Algebra and $x \in \mathcal{A}$ be an element that has no inverse. Prove that x belongs to a non-trivial maximal ideal.

Solution: Consider the set

$$\mathcal{I} := \{y \in \mathcal{A} : y = xz, z \in \mathcal{A}\} \ .$$

Obviously $x \in \mathcal{I}$. This set is an algebra; if $y_1, y_2 \in \mathcal{I}$ so is its sum and scalar multiples and also $y_1 y_2$. This set is an ideal. If $u \in \mathcal{A}$ and $y \in \mathcal{I}$ then $y = xz$ and hence $uy = uzx \in \mathcal{I}$. This ideal is not the zero ideal. Moreover, it is not all of $\mathcal{A}$, because unless $x = 0$ there exists $\lambda \in \mathbb{C}$ such that $\lambda e - x$ is invertibel. However all the elements in $\mathcal{I}$ are not invertible. Hence $\mathcal{I}$ is a proper ideal and we know from the class than any proper ideal is contained in a proper maximal ideal.

III: Again, $\mathcal{A}$ is a commutative Banach Algebra. Prove that closed ideal $\mathcal{I} \subset \mathcal{A}$ is a proper subset of a nontrivial ideal if and only if its factor algebra $\mathcal{A}/\mathcal{I}$ has nontrivial ideals.

Solution: Suppose that $\mathcal{I} \subset \mathcal{J} \subset \mathcal{A}$, $\mathcal{I} \neq \mathcal{J}$ and $\mathcal{J} \neq \mathcal{A}$. In the equivalence class $[x]_{\mathcal{I}} = x + \mathcal{I}$ take any two elements and declare them as equivalent if their difference is in $\mathcal{J}$. This defines a non-trivial ideal in the factor algebra $\mathcal{A}/\mathcal{I}$. The converse is similar.

IV: Show that the Gelfand transform is linear, multiplicative, and that $\widehat{e} = 1$.

Solution: recall the definition of the Gelfand transform. There is a one-to-one correspondence between maximal ideals and multiplicative functionals. Let M be such a maximal ideal and f_M the corresponding multiplicative functional. The Gelfand transform is the map that associates for any $x \in \mathcal{A}$ the number $f_M(x)$. We have to consider this as a function on the compact space of maximal ideals. We denote this map by $\gamma(x)$, i.e., $\gamma(x) = f_M(x)$. Again, for every $x \in \mathcal{A}$ $\gamma(x)$ is a function on the space of maximal ideals of $\mathcal{A}$. Now for $x, y \in \mathcal{A}$ and any M we have that

$$\gamma(xy) = f_M(xy) = f_M(x) f_M(y) = \gamma(x)\gamma(y) \ .$$

moreover $\gamma(e) = f_M(e) = 1$, since f_M is a multiplicative functional.

V : Let e be the unit element in a C^* algebra. Prove that $e^* = e$ and that $\|e\| = 1$

For any element $x \in \mathcal{A}$, $e^* x = (x^* e)^* = (x^*)^* = x$ and since the unit element is unique, $e^* = e$. Since $\mathcal{A}$ is a C^* algebra we have that $\|e\| = \|e^* e\| = \|e\|^2$ and hence $\|e\| = 1$.