Homework 4, due Wednesday April 5

I: Let x be an element of a Banach algebra $\mathcal{A}$ with in involution $x \to x^*$. Assume that for all $x \in \mathcal{A}$, $\|x\|^2 \le \|xx^*\|$. Show that $\mathcal{A}$ is a C^* algebra.

Solution: We have that

$$\|x\|^2 \le \|xx^*\| \le \|x\|\|x^*\|$$

and hence $\|x\| \le \|x^*\|$. Similarly we find that $\|x^*\| \le \|x\|$ and hence $\|x\| = \|x^*\|$. Thus, $\|x\|^2 \le \|xx^*\| \le \|x\|\|x^*\| = \|x\|^2$ and $\mathcal{A}$ is a C^* algebra.

II: Consider the set $\mathcal{A}$ of continuous complex valued functions on the closed disk $D = \{z : |z| \le 1\}$ in the complex plane that are analytic in the interior of D. Let $\|f\| = \sup_{|z|\le 1} |f(z)|$. Show that $\mathcal{A}$ is a commutative Banach algebra with unit. (Hint: Cauchy's theorem might come in handy).

Solution: That these functions form an algebra is obvious. We have to show that $\|f\|$ defines a norm. It is clear that if $\|f\| = 0$ the $f = -0$. That $\|\lambda f\| = |\lambda|\|f\|$ and that $\|f+g\| \le \|f\| + \|g\|$ is obvious. Likewise, $\sup_{|z|\le 1} |f(z)g(z)| \le \sup_{|z|\le 1} |f(z)| \sup_{|z|\le 1} |g(z)|$ which shows that $\|fg\| \le \|f\|\|g\|$. The unit element is given by the function which is identically equals to 1. It remains to show completeness. Let f_n be a Cauchy sequence. This means that the sequence is a uniform Cauchy sequence on the closed unit disk and hence converges uniformly on the closed disk to a function $\tilde{f}$ which is continuous on the closed disk. We have to show that this function is analytic. Pick any z in the interior of D. By Cauchy's theorem we have that

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\zeta)}{z - \zeta} d\zeta$$

where C is the boundary of the unit disk. Define

$$f(z) = \frac{1}{2\pi i} \int_C \frac{\tilde{f}(\zeta)}{z - \zeta} d\zeta$$

which is analytic in the interior of D and continuous on D. Moreover,

$$|f(z) - f_n(z)| \le \frac{1}{\mathrm{dist(z, C)}} \sup_{|\zeta|=1} |\tilde{f}(\zeta) - f_n(\zeta)| \to 0$$

and hence $f(z) = \tilde{f}(z)$ which is hence analytic in the interior.

III: With the same setup as in the previous problem, show that $f^*(z) = \overline{f(\overline{z})}$ defines an isometric involution in $\mathcal{A}$.

Solution: The function $f^*(z)$ is analytic in the interior and hence $*$ defines an involution. It is isometric, since by the maximum principle

$$\|f\| = \max_{|z|=1} |f(z)|$$

and

$$\|f^*\| = \max_{|z|=1} |\overline{f}(\overline{z})| = \max_{|z|=1} |f(\overline{z})| = \max_{|z|=1} |f(z)| = \|f\| \ .$$

IV: Let $\mathcal{H}$ be a Hilbert space and $L(\mathcal{H})$ with the operator norm is the space of bounded operators. Show that $L(\mathcal{H})$ is a (non-commutative) C* algebra.

Solution: Recall that the set bounded linear operators from a normed space to a Banach space forms itself a Banach space. Moreover, $\|AB\| \leq \|A\|\|B\|$. The only condition not completely obvious is that

$$\|A\|^2 = \sup_{\|x\|=1} \|Ax\|^2 = \sup_{\|x\|=1} \langle x, A^*Ax \rangle = \|AA^*\|$$

since A^*A is self adjoint.

V : Show that the space of compact operators form a closed subalgebra without unit of $L(\mathcal{H})$.

Solution: This follows from the fact that the limit of a sequence of compact operators that converge in the operator norm is again a compact operator. Moreover the identity operator is not a compact operator if the Hilbert is not finite dimensional.