C^* ALGEBRAS

1. The commutative case

We shall assume that $\mathcal{A}$ is a Banach Algebra. A map $\star : \mathcal{A} \to \mathcal{A}$ is called an **involution** if $(x^*)^* = x$, $\lambda x + \ muy)^* = \overline{\lambda} x^* + \overline{\mu} y^*$ and $(xy)^* = y^* x^*$. An element x with $x = x^*$ is called **self adjoint**. If $x \in \mathcal{A}$ define the self adjoint elements

$$y = \frac{1}{2}(x + x^*) \ , \ z = \frac{1}{2i}(x - x^*)$$

and note that $x = y + iz$.

Definition 1.1. *An algebra with involution is called* **symmetric** *if and only if for every $x \in \mathcal{A}$, $\widehat{x^*}(\mathcal{M}) = \overline{\widehat{x}(\mathcal{M})}$.*

Definition 1.2. *A Banach Algebra with an involution is called a C^* algebra if and only if*

$$\|xx^*\| = \|x\|\|x^*\| \ .$$

Theorem 1.3. *Gelfand-Naimark If $\mathcal{A}$ is a commutative C^* algebra, then the Gelfand Transform is an isometric isomorphism between the algebra $\mathcal{A}$ and $C(\mathcal{M}(\mathcal{A}))$. Moreover,we have that*

$$(\widehat{x}(\mathcal{M}))^* = \overline{\widehat{x}(\mathcal{M})} = \widehat{x^*}(\mathcal{M}) \ .$$

Proof. Recall that a Banach Algebra that is symmetric and regular is isometrically isomorphic to the algebra of continuous functions over the compact space of maximal ideals. Hence, we show that $\mathcal{A}$ is symmetric and regular. Let $x \in \mathcal{A}$ and note that

$$\|(xx^*)^2\| = \|(xx^*)(xx^*)^*\| = \|xx^*\|^2 = \|x\|^2\|x^*\|^2 \ .$$

Further

$$\|x^2(x^*)^2\| = \|x^2(x^2)^*\| = \|x^2\|\|(x^*)^2\|$$

and hence

$$\|x\|^2\|x^*\|^2 = \|x^2\|\|(x^*)^2\| \leq \|x\|^2\|x^*\|^2 \ .$$

Thus, $\|x^2\| = \|x\|^2$ and the algebra is regular. We show that the algebra is symmetric. Let $f \in \mathcal{A}^*$ be a multiplicative functional. If $x \in \mathcal{A}$ with $x = x^*$ we shall show that $f(x)$ is real. Suppose not. Then $f(x) = a + ib$ with $b \neq 0$. This means that the element $y = \frac{1}{b}(x - ae)$ satisfies $f(y) = i$. Hence, $y - ie$ is not invertible and hence $(h - ie)^* = h + ie$ is not invertible either and there exists a multiplicative functional f_0 with $f_0(h) = -i$. For any number $c \in \mathbb{R}$ we have that

$$f(h + ice) = (1 + c)i \ , f_0(h - ice) = -(1 + c)i \ .$$

Thus, we have that $1 + c \leq |f(h + ice)| \leq \|h + ice\|$ and similarly $1 + c \leq \|h - ice\|$. Hence

$$\|h^2 + t^2 e\| = \|(h + ice)(h + ice)^*\| = \|(h + ice)\|\|(h + ice)^*\| \geq (1 + t)^2$$

and since $\|h^2 + t^2 e\| \leq \|h^2\| + t^2$ this leads to a contradiction for t large. Hence, $f(x) \in \mathbb{R}$ for all self adjoint elements. Hence, for a general $x \in \mathcal{A}$ we have

$$f(x) = f(y + iz) = f(y) + if(z) \ , f(x^*) = f(y - iz) = f(y) - if(z) = \overline{f(x)}$$

because y, z are self adjoint. Hence, $\mathcal{A}$ is symmetric. $\square$

2. Application to spectral theory

We apply this theorem to spectral theory and follow closely the text of Edelman, Milman and Tsolomitis. Consider the set of all pairwise commuting normal operators on a Hilbert space $\mathcal{H}$. An operator A is normal if $AA^* = A^*A$. By considering polynomials $p(w,z)$ in two variables, we see that expressions of the form $p(A, A^*)$ generate an algebra. The norm on this algebra is the standard operator norm and we may take the closure of the algebra with respect to that norm. This yields a commutative Banach Algebra $\mathcal{A}$. For any element $A \in \mathcal{A}$ we have that

$$\|A\|\|A^*\| = \|AA^*\| \ .$$

This follows from the fact that AA^* is self adjoint and hence

$$\|AA^*\| = \sup_{\|x\|=1} (x, AA^*x) = \sup_{\|x\|=1} \|A^*x\|^2 = \|A^*\|^2$$

and the fact that $\|A^*\| = \|A\|$. Hence, $\mathcal{A}$ is a C^* algebra. Recall, that the Gelfand Transform is an isometric isomorphism between $\mathcal{A}$ and $C(\mathcal{M}(\mathcal{A}))$. Hence, for every function $f \in C(\mathcal{M}(\mathcal{A}))$ there exists and element $T_f : \mathcal{H} \to \mathcal{H}$. The map $f \to T_f$ is linear and multiplicative, i.e., $T_{f+g} = T_f + T_g$, $T_{\lambda f} = \lambda T_f$ and $T_{fg} = T_f T_g$. Moreover, $T_f^* = T_{\overline{f}}$, in particular for f real, $T_f^* = T_f$ and T_f is self adjoint. As a consequence for $f \geq 0$ we have that

$$T_f = T_{\sqrt{f}} T_{\sqrt{f}} = T^*_{\sqrt{f}} T_{\sqrt{f}} \geq 0$$

as operator on $\mathcal{H}$. Pick any two vectors $x, y \in \mathcal{H}$ and form

$$\langle T_f x, y\rangle \ .$$

For x, y fixed, the functional $f \to \langle T_f x, y\rangle$ is bounded and linear as a functional on $C(\mathcal{M}(\mathcal{A}))$ and hence there exists a regular signed Borel measure such that

$$\langle T_f x, y\rangle = \int_{\mathcal{M}} f d\mu_{x,y} \ .$$

we work out some properties of this measure. We define

$$\|\mu_{x,y}\| = \sup_{\|f\|=1} |\int_{\mathcal{M}} f d\mu_{x,y}| \ .$$

Hence

$$\|\mu_{x,y}\| \leq \|T_f\|\|x\|\|y\| = \|f\|\|x\|\|y\|$$

so that

$$\|\mu_{x,y}\| \leq \|x\|\|y\| \ .$$

Further

$$\mu_{x,y} = \overline{\mu_{y,x}}$$

which follows from

$$\overline{\langle T_f y, x\rangle} = \langle x, T_f y\rangle = \langle T_f^* x, y\rangle = \langle T_{\overline{f}} x, y\rangle \ .$$

We also have the polarization identity

$$\langle T_f x, y\rangle =$$

$$\frac{1}{4}\left[\left(\langle T_f(x+y),(x+y)\rangle - \langle T_f(x-y),(x-y)\rangle\right) - i\left(\langle T_f(x+iy),(x+iy)\rangle - \langle T_f(x-iy),(x-iy)\rangle\right)\right]$$

thus, in terms of the measure we have

$$\mu_{x,y} = \frac{1}{4}\left[(\mu_{x+y,x+y} - \mu_{x-y,x-y}) - i\,(\mu_{x+iy,x+iy} - \mu_{x-iy,x-iy})\right] \ .$$

The measure is linear in x and conjugate linear in y. For $f \geq 0$ we have that

$$|\langle T_f x, y\rangle| = \langle T_{\sqrt{f}}x, T_{\sqrt{f}}y\rangle \leq \|T_{\sqrt{f}}x\|\|T_{\sqrt{f}}y\|$$

so that

$$|\langle T_f x, y\rangle|^2 \leq \langle T_f x, x\rangle\langle T_f y, y\rangle \ .$$

In terms of μ this reads

$$|\int_{\mathcal{M}} f d\mu_{x,y}|^2 \leq \int_{\mathcal{M}} f d\mu_{x,x} \int_{\mathcal{M}} f d\mu_{y,y} \ .$$

Once more, we gat as a consequence the previous inequality, because

$$\int_{\mathcal{M}} f d\mu_{x,x} \leq \|f\| \int_{\mathcal{M}} d\mu_{x,x} = \|f\|\|x\|^2 \ .$$

The measure $\mu_{x,x}$ is positive, since for $f \geq 0$, $\langle T_f x, x\rangle = \|T_{\sqrt{f}}x\|^2$. The goal is to define T_χ where χ is a bounded Borel function. For such a χ we may define

$$\int_{\mathcal{M}} \chi d\mu_{x,x}$$

which makes sense, because the measure is positive. In particular we have that

$$\int_{\mathcal{M}} \chi_E d\mu_{x,x} \leq \|\chi\|_\infty \int_{\mathcal{M}} d\mu_{x,x} = \|\chi\|_\infty \|x\|^2 \ .$$

Using the polarization identity the following integral is defined

$$\int_{\mathcal{M}} \chi d\mu_{x,y} \ .$$

It is linear in x and conjugate linear in y and bounded. Hence there exists a bounded operator which we denote by T_χ such that

$$\langle T_\chi x, y\rangle = \int_{\mathcal{M}} \chi d\mu_{x,y} \ .$$

We have to check a number of properties:

$$a) \ \ T_{\overline{\chi}} = T^*_\chi$$

$$b) \ \ T_{\chi_1\chi_2} = T_{\chi_1}T_{\chi_2}$$

c) If $S \in L(\mathcal{H}), ST_f = T_f S$ for all $f \in C(\mathcal{M})$, then $ST_\chi = T_\chi S$.

The following comes in handy.

Lemma 2.1. *let χ be a Borel function and let $\varepsilon > 0$ be given. There exists a function $f \in C(\mathcal{M})$ such that for all $x \in \mathcal{H}$*

$$\int_{\mathcal{M}} |\chi - f| d\mu_{x,x} \leq \varepsilon \|x\|^2 \ .$$

This follows from the regularity of Borel measures.

Corollary 2.2. *For every $x, y \in mathcalH$ and $\varepsilon > 0$ there exists $f \in C(\mathcal{M})$, such that*

$$| \int_{\mathcal{M}} \chi d\mu_{x,y} - \int_{\mathcal{M}} f d\mu_{x,y}| < \varepsilon \|x\| \|y\| \ .$$

With the help of these approximations the statements a), b) and c) are evident.

Corollary 2.3. *Let χ be the characteristic function of a Borel set. Then*

$$T_\chi^2 = T_\chi$$

and

$$T_\chi^* = T_\chi$$

i.e., T_χ is a self adjoint projection.

Proof.

$$T_\chi^2 = T_\chi T_\chi = T_{\chi^2} = T_\chi \ ,$$

and

$$T_\chi^* = T_{\overline{\chi}} = T_\chi \ .$$

□

3. Cyclic vectors

Pick any $x \in H$ and consider the space spanned by

$$T_f x \ , f \in C(\mathcal{M}) \ .$$

The closure of this set form a subspace $\mathcal{H}_x$ of $\mathcal{H}$. We define a map U from $C(\mathcal{M})$ to $\mathcal{H}_x$ by setting

$$Uf = T_f x \ .$$

The map is clearly linear and we have that

$$\|Uf\|^2 = \|T_f x\|^2 = \langle x, T^* f T_f x \rangle = \langle x T_{|f|^2} x \rangle = \int_{\mathcal{M}} |f|^2 \mu_{x,x} \ .$$

Hence the map U is unitary from $L^2(d\mu_{x.x})$ to the Hilbert space $\mathcal{H}_x$. In particular for T_g we have that

$$U^* T_g U f = U^* T_g T_f x = U^* T_{gf} x = U^* U(gf) = gf$$

and the operator T_g is unitarily equivalent to multiplication by g on $L^2(d\mu_{x,x})$. Next, consider $y \perp \mathcal{H}_x$ this leads to a Hilbert space $\mathcal{H}_y$ and the same considerations apply. A simple argument using Zorn's lemma leads to the existence of a unitary

$$U : \oplus_{\alpha \in I} L^2(\mathcal{M}, \mu_{x_\alpha, x_\alpha}) \to \mathcal{H}$$

so that the restriction of any element in $\mathcal{A}$ to $\mathcal{H}_{x_\alpha}$ is unitarily equivalent to a multiplication operator.

4. Non-commutative C^* algebras

One knows from operator theory that for any two linear operators x, y the spectrum of xy is the same as the spectrum of yx. This remains true in an abstract setting.

Lemma 4.1. *Let $\mathcal{A}$ be a C^* algebra with unit element e. Then for any two elements $x, y \in \mathcal{A}$ $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.*

Proof. The proof is simple and left as an exercise. □