A SYNOPSIS OF HILBERT SPACE THEORY

Below is a summary of Hilbert space theory that you find in more detail in any book on functional analysis, like the one Akhiezer and Glazman, the one by Kreiszig or the one by Eidelman, Milman and Tsolomitis. These notes follow Akhiezer and Glazman.

Definition 0.1. Inner product space. *An inner product space $\mathcal{R}$ is a linear space endowed with a function $(\cdot,\cdot) : \mathcal{R} \times \mathcal{R} \to \mathbb{C}$ which has the following properties*

a) $(f, \alpha g + \beta h) = \alpha(f, g) + \beta(f, h)$ all $f, g, h \in \mathcal{R}$ and all $\alpha, beta \in \mathbb{C}$,

b) $(f, g) = \overline{(g, f)}$ all $f, g \in \mathcal{R}$,

c) $(f, f) \geq 0$ and 0 only if $f = 0$.

We set $\|f\| = \sqrt{(f, f)}$.

A simple consequence is

Lemma 0.2. Schwarz.s inequality and triangle inequality. *For every $f, g \in \mathcal{R}$ we have that*

$$|(f, g)| \leq \|f\|\|g\| \tag{1}$$

and

$$\|f + g\| \leq \|f\| + \|g\| \ . \tag{2}$$

Proof. We may assume that neither f nor g are zero and we may also assume that $(f, g) \neq 0$ for otherwise the inequality (1) is obvious. Set

$$X = \frac{f}{\|f\|} \ , \ Y = \frac{g}{|g\|}$$

and

$$e^{i\phi} = \frac{(X, Y)}{|(X, Y)|} \ .$$

The inequality (1) is equivalent to

$$|(X, Y)| \leq 1 \ .$$

Now

$$0 \leq \|e^{i\phi}X + Y\|^2 = 2 + e^{-i\phi}(X, Y) + e^{i\phi}(Y, X) = 2 + 2|(X, Y)|$$

and likewise

$$0 \leq \|e^{i\phi}X - Y\|^2 = 2 - e^{-i\phi}(X, Y) - e^{i\phi}(Y, X) = 2 - 2|(X, Y)|$$

from which the result follows. The triangle inequality (2) is now a simple consequence of Schwarz's inequality, because

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2 + (f, g) + (g, f) \leq \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| = (\|f\| + \|g\|)^2 \ .$$

□

Lemma 0.3. Parallelogram identity. *For any two vectors $f, g \in \mathcal{R}$ we have that*

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2 \ .$$

Proof. A simple computation. □

Remark 0.4. *Note that $\mathcal{R}$ endowed with the norm $\|\cdot\|$ is a metric space. The distance between two vectors f, g is given by $\|f-g\|$. Hence we know what the open and closed sets are. A set $S \subset \mathcal{R}$ is open if for any $f \in S$ there exists $\varepsilon > 0$ so that the ball*

$$B_\varepsilon(f) = \{g \in \mathcal{R} : \|f-g\| < \varepsilon\}$$

is a subset of S. A subset $T \subset \mathcal{R}$ is closed if its complement in $\mathcal{R}$, T^c is open. The following statement is easy to prove: A subset $T \subset \mathcal{R}$ is closed if and only if T contains all its limit points. More precisely, if $f^{(k)}$ is any sequence in T which converges to some element $f \in \mathcal{R}$, then $f \in T$.

Definition 0.5. Completeness. *A inner product space is complete if for every Cauchy Sequence $f^{(k)} \in \mathcal{R}$ there exists $f \in \mathcal{R}$ such that $f^{(k)}$ converges to f.*

Recall that the statement "$f^{(k)}$ converges to f" means that for any $\varepsilon > 0$ there exists a positive integer N so that for all $k > N$, $\|f^{(k)} - f\| < \varepsilon$. Also recall that the statement "$f^{(k)}$ is a Cauchy Sequence" means that for any $\varepsilon > 0$ there exists a positive integer N so that $\|f^{(k)} - f^{(\ell)}\| < \varepsilon$ for all $k, \ell > N$.

Definition 0.6. Hilbert space. *A complete inner product space is called a "Hilbert" space and we denote it by $\mathcal{H}$.*

A set $C \subset \mathcal{H}$ is convex, if with $f, g \in C$, $\lambda f + (1-\lambda)g \in C$ for all $0 \le \lambda \le 1$.

Theorem 0.7. Minimial distance. *Let C be a closed convex subset of a Hilbert space $\mathcal{H}$ and $f \in \mathcal{H}$ arbitrary. There exists a unique vector $h \in C$ such that*

$$\|f-h\| = d := \inf\{\|f-g\| : g \in C\} \ .$$

Proof. By the definition of the infimum, for any positive integer n there exists $g^{(n)} \in C$ such that

$$d \le \|f - g^{(n)}\| \le d + \frac{1}{n} \ .$$

Now for any positive integer n, m, by the parallelogram identity,

$$\|\frac{f-g^{(n)}}{2} + \frac{f-g^{(m)}}{2}\|^2 + \|\frac{f-g^{(n)}}{2} - \frac{f-g^{(m)}}{2}\|^2 = 2\|\frac{f-g^{(n)}}{2}\|^2 + 2\|\frac{f-g^{(m)}}{2}\|^2 \ ,$$

which can be rewritten as

$$\|f - \frac{g^{(n)}+g^{(m)}}{2}\|^2 + \|\frac{g^{(n)}-g^{(m)}}{2}\|^2 = 2\|\frac{f-g^{(n)}}{2}\|^2 + 2\|\frac{f-g^{(m)}}{2}\|^2 \ ,$$

Since C is convex, $\frac{g^{(n)}+g^{(m)}}{2} \in C$ and

$$d^2 \le \|f - \frac{g^{(n)}+g^{(m)}}{2}\|^2 + \|\frac{g^{(n)}-g^{(m)}}{2}\|^2 \le \frac{1}{2}(d+\frac{1}{n})^2 + \frac{1}{2}(d+\frac{1}{m})^2$$

or

$$\|\frac{g^{(n)}-g^{(m)}}{2}\|^2 \le d(\frac{1}{n} + \frac{1}{m}) + \frac{1}{2n^2} + \frac{1}{2m^2} \ .$$

Pick a positive integer N so that

$$4d(\frac{1}{n}+\frac{1}{m})+\frac{2}{n^2}+\frac{2}{m^2}<\varepsilon^2$$

and we learn that whenever $n, m > N$,

$$\|g^{(n)}-g^{(m)}\|<\varepsilon\ .$$

Hence, $g^{(n)}$ is a Cauchy Sequence and since $\mathcal{H}$ is complete, there exists $h \in \mathcal{H}$ so that $g^{(n)}$ converges to h. Because C is closed, $h \in C$. We have to show that $\|f-h\| = d$. This follows easily, since

$$d \le \|f-h\| \le \|f-g^{(n)}\|+\|g^{(n)}-h\| \le d+\frac{1}{n}+\|g^{(n)}-h\|\ .$$

Suppose now that $h_1, h_2 \in C$ are two vectors with

$$d = \|f-h_1\| = \|f-h_2\|\ .$$

Then, again by the parallelogram identity

$$\|f-\frac{h_1+h_2}{2}\|^2+\|\frac{h_1-h_2}{2}\|^2 = 2\|\frac{f-h_1}{2}\|^2+2\|\frac{f-h_2}{2}\|^2 = d^2\ .$$

Hence

$$\|\frac{h_1-h_2}{2}\|^2 = d^2-\|f-\frac{h_1+h_2}{2}\|^2 \le 0$$

since $\frac{h_1+h_2}{2} \in C$. □

Definition 0.8. Linear manifold, subspace *A subset $M \subset \mathcal{H}$ is a linear manifold if it is closed under addition of vectors and scalar multiplication. If a linear manifold $G \subset \mathcal{H}$ is closed, then it is a complete inner product space, i.e., a Hilbert space. In this case we call G a subspace of $\mathcal{H}$.*

We say that two vectors g, h are **orthogonal or perpendicular** to each other if $(f, g) = 0$.

Theorem 0.9. Orthogonal complement *Let G be a subspace of $\mathcal{H}$. Then for any $f \in \mathcal{H}$ there exits two uniquely specified vectors g and h such that $f = g + h$, $g \in G$ and $h \perp G$, i.e., h is perpendicular to every vector in G.*

Proof. The subspace G is convex (since it is linear) and closed. Hence there exists $g \in G$ so that

$$\|f-g\| = \inf\{\|f-u\| : u \in G\}\ .$$

We show that $f - g$ is perpendicular to every vector in G. Pick $v \in G$ and consider the vector $g + tv$ where t is an arbitrary complex number. Since $g + tv \in G$ we have that

$$\begin{aligned} d^2 \le |f-g-tv\|^2 &= \|f-g\|^2+\bar{t}(f-g,u)+t(u,f-g)+|t|^2\|u\|^2 \\ &= d^2+\bar{t}(f-g,u)+t(u,f-g)+|t|^2\|u\|^2 \end{aligned}$$

Hence we have for all $t \in \mathbb{C}$

$$t(f-g,u)+\bar{t}(u,f-g)+|t|^2\|u\|^2 \ge 0\ .$$

Choosing t real positive we have that

$$\Re(f-g,u)+t\|u\|^2 \ge 0$$

and since $t > 0$ is arbitrary we find $\Re(f-g,u) \ge 0$. Likewise, choosing $t < 0$ we find that $\Re(f-g,u) \le 0$ and hence $\Re(f-g,u) = 0$. Next we choose t to be purely imaginary and find, by a similar reasoning that $\Im(f-g,u)=0$. Thus $(f-g,u)=0$ and since $u \in G$ is arbitrary $f-g \perp G$. If $f = g_1+h_1 = g_2+h_2$ are two such decomposition, we have that

$$g_1 - g_2 = h_2 - h_1 \ .$$

Because $g_1 - g_2 \in G$ and $h_2 - h_1 \perp G$ we must have that $g_1 - g_2 = 0$ and hence $h_1 = h_2$. □

Definition 0.10. Bounded linear functionals *A bounded linear function is a function $\ell : \mathcal{H} \to \mathbb{C}$ that satisifes $\ell(f+g) = \ell(f) + \ell(g)$ for all $f, g \in \mathcal{H}$ and $\ell(\alpha f) = \alpha \ell(f)$ all $f \in \mathcal{H}$ and $\alpha \in \mathbb{C}$. Bounded means that there exists a positive constant C such that*

$$|\ell(f)| \le C\|f\|$$

for all $f \in \mathcal{H}$.

Proposition 0.11. *A linear functional is bounded if and only if it is continuous.*

Proof. If ℓ is bounded then for any $f, g \in \mathcal{H}$

$$|\ell(f) - \ell(g)| = |\ell(f-g)| \le C\|f-g\|$$

from which the continuity is an immediate consequence. Suppose now that ℓ is continuous. This mean that for any $\varepsilon > 0$ there exists $\delta > 0$ so that

$$|\ell(f) - \ell(g)| < \varepsilon$$

whenever $\|f-g\| < \delta$. Hence, whenever $\|f\| < \delta$ then

$$|\ell(f)| < \varepsilon \ .$$

Now, pick any $f \in \mathcal{H}$ and consider

$$g = \frac{\delta}{2}\frac{f}{\|f\|} \ .$$

Then $\|g\| = \frac{\delta}{2} < \delta$ and

$$\ell(f) = \ell(g)\frac{2\|f\|}{\delta}$$

and hence

$$|\ell(f)| < \frac{2\varepsilon}{\delta}\|f\| \ .$$

□

Theorem 0.12. Riesz representation theorem *For any bounded linear functional ℓ on a Hilbert space $\mathcal{H}$ there exists a unique vector $v \in \mathcal{H}$ so that*

$$\ell(f) = (v, f)$$

for all $f \in \mathcal{H}$.

Proof. Consider the set

$$G = \{f \in \mathcal{H} : \ell(f) = 0\}$$

This set is a subspace G of the Hilbert space $\mathcal{H}$. Either the function ℓ is the trivial functional in which case $v = 0$ or there exists $g \in \mathcal{H}$ with $\ell(g) \ne 0$. By the projection lemma there exists a unique $u \perp G$ and $w \in G$ such that $g = u + w$ and $\ell(u) = \ell(g) \ne 0$. For $f \in \mathcal{H}$ arbitrary we have that $\ell(f)u - \ell(u)f \in G$ and by taking the inner product with u we find

$$\ell(f)(u,u) - \ell(u)(u,f) = 0$$

or

$$\ell(f) = \frac{\ell(u)}{(u,u)}(u,f)$$

and if we set

$$v = \frac{\overline{\ell(u)}}{(u,u)}u$$

we have that

$$\ell(f) = (v,f) \text{ for all } f \in \mathcal{H} \ .$$

If v_1, v_2 are two such functions then

$$(v_1 - v_2, f) = 0$$

for all $f \in \mathcal{H}$ and hence $v_1 = v_2$. □

We continue with an interesting topic that runs under the name "uniform boundedness principle". We deal with this in a different fashion than usual.

Consider a function

$$p : \mathcal{H} \to \mathbb{R} \ .$$

Recall tat p is continuous at $f_0 \in \mathcal{H}$ if for any $\varepsilon > 0$ there exist δ such that

$$\varepsilon > p(f) - p(f_0) > -\varepsilon$$

whenever $\|f - f_0\| < \delta$.

Note that the above definition has two parts: We say that p is **lower semi continuous at** f_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ so that

$$p(f) - p(f_0) > -\varepsilon$$

whenever $\|f - f_0\| < \delta$. It is **upper semi continuous** if

$$\varepsilon > p(f) - p(f_0)$$

whenever $\|f - f_0\| < \delta$.

It is easy to see that p is lower semi continuous if the set $\{f \in \mathcal{H} : p(f) > t\}$ is open for any $t \in \mathbb{R}$. Likewise, p is upper semi continuous if for all $t \in \mathbb{R}$ the set $\{f \in \mathcal{H} : p(f) < t\}$. In general, there is no reason why a lower semi continuous function is continuous but there is an interesting exception.

Definition 0.13 (Subadditive functions)**.** *A function* $p : \mathcal{H} \to \mathbb{R}$ *is a* **seminorm** *if*

a) $p(f+g) \le p(f) + p(g)$ all $f, g \in \mathcal{H}$

b) $p(\alpha f) = |\alpha| p(f)$.

Clearly, $p(0) = 0$ and $p(-f) = f$. Hence, $p(f) \ge 0$ for all $f \in \mathcal{H}$ since

$$0 = p(0) = p(f - f) \le 2p(f) \ .$$

Note that $p(f)$ is almost a norm, except that $p(f) = 0$ does not entail that $f = 0$. Note, that the book calls such a function 'convex', which in its usual use has a different definition.

The following theorem is now of great interest. It is crucial that $\mathcal{H}$ is complete.

Theorem 0.14. *Any lower semi continuous seminorm $p : \mathcal{H} \to \mathbb{R}$ is bounded, i.e., there is a constant M such that*

$$p(f) \le M\|f\| \ .$$

Proof. Given an open ball in the Hilbert space, $B_\rho(f)$. We shall prove that p is bounded on $B_\rho(f)$ if an only if it is bounded on the open unit ball, $B_1(0)$ centered at the origin. Assume that p is bounded on $B_1(0)$, i.e., $p(g) \le M$ for all g with $\|g\| < 1$. Pick any $h \in B_\rho(f)$. This means that $\|f-h\| < \rho$ or

$$\|\frac{f-h}{\rho}\| < 1 \ .$$

Hence

$$M \ge p(\frac{f-h}{\rho}) = \frac{1}{\rho} p(f-h)$$

and for all $h \in B_\rho(f)$

$$p(h) = p(h-f+f) \le p(h-f) + p(f) \le M\rho + p(f) \ .$$

Conversely, if $p(h) \le C$ for all $h \in B_\rho(f)$, then for $g \in B_1(0)$ we can write

$$g = \frac{h-f}{\rho}$$

where $f = h - g\rho$. Now

$$p(g) = p(\frac{h-f}{\rho}) = \frac{p(f-h)}{\rho} \le \frac{p(f)+p(h)}{\rho} \le \frac{2C}{\rho} \ .$$

Assume that p is not bounded in $B_1(0)$. Hence it is unbounded on every ball. There exists $f_1 \in B_1(0)$ such that

$$p(f_1) > 1 \ .$$

Since p is lower semi continuous, there exists $\rho_1 < 1/2$ such that $p(f) > 1$ on the ball $B_{\rho_1}(f_1)$. Since p is not bounded on $B_{\rho_1}(f_1)$, there exists $f_2 \in B_{\rho_1}(f_1)$ with $p(f_2) > 2$. Again, since p is lower semi continuous, there exists $\rho_2 < \rho_1/2$ so that $p(f) > 2$ on all of $B_{\rho_2}(f_2)$. Continuing in this fashion we have a sequence of balls

$$B_{\rho_1}(f_1) \supset B_{\rho_2}(f_2) \supset B_{\rho_3}(f_3) \cdots$$

with $\rho_k < \rho_{k-1}/2$ so that $p(f) > k$ for all $f \in B_{\rho_k}(f_k)$. Since f_k is a Cauchy Sequence and $\mathcal{H}$ is complete, there exists f with $\lim_{k\to\infty} f_k = f$. Further, since $f \in B_{\rho_k}(f_k)$ for all k we have that $p(f) > k$ for all k which is not possible.

□

Remark 0.15. *To see the role played by lower semi-continuity let us look at the 'argument' that does not use this assumption: Assume that p is not bounded in $B_1(0)$. Hence it is unbounded on every ball. There exists $f_1 \in B_1(0)$ such that*

$$p(f_1) > 1 \ .$$

Pick $\rho_1 < 1/2$ so that $B_{\rho_1}(f_1) \subset B_1(0)$. (Note, that we do not know whether $p(f) > 1$ for all $f \in B_{\rho_1}(f_1)$). Since p is not bounded on $B_{\rho_1}(f_1)$ there exists $f_2 \in B_{\rho_1}(f_1)$ with $p(f_2) > 2$. Pick $\rho_2 < \rho_1/2$ There exists $f_3 \in B_{\rho_2}(f_2)$ with $p(f_3) > 3$ and so on. We then have

$$B_{\rho_1}(f_1) \supset B_{\rho_2}(f_2) \supset B_{\rho_3}(f_3) \cdots$$

and $p(f_k) > k$. The sequence f_k is a Cauchy sequence and hence converges to some f. Note that at this stage we cannot say anything about the value of $p(f)$.

More popular than the preceding theorem is the following corollary.

Theorem 0.16 (Principle of uniform boundedness)**.** *Let $\{\ell_\alpha\}_{\alpha \in J}$ be a family of linear functionals on a Hilbert space $\mathcal{H}$ such that for any fixed $f \in \mathcal{H}$ we have that*

$$\sup_{\alpha \in J} |\ell_\alpha(f)| \leq C_f \ .$$

Then there exists a constant M such that

$$\sup_{\alpha \in J} \|\ell_\alpha\| \leq M \ .$$

Proof. Consider the function

$$p(f) = \sup_{\alpha \in J} |\ell_\alpha(f)|$$

which is defined and hence finite for every $f \in \mathcal{H}$. We have that

$$p(\lambda f) = |\lambda| p(f) \ , \lambda \in \mathbb{C} \ , f \in \mathcal{H}$$

and

$$p(f+g) = \sup_{\alpha \in J} |\ell_\alpha(f+g)| \leq \sup_{\alpha \in J} \{|\ell_\alpha(f)| + |\ell_\alpha(g)|\} \leq p(f) + p(g) \ .$$

Hence, p is subadditive. We shall show that p is lower semi-continuous. To prove this, pick $\varepsilon > 0$ arbitrary. For any $f_0 \in \mathcal{H}$ we have to find $\delta > 0$ so that

$$p(f) > p(f_0) - \varepsilon \ .$$

There exists $\alpha \in J$ so that

$$p(f_0) < |\ell_\alpha(f_0)| + \varepsilon/2 \ ,$$

and hence

$$p(f) \geq |\ell_\alpha(f)| \geq |\ell_\alpha(f_0)| - |\ell_\alpha(f - f_0)| > p(f_0) - \varepsilon/2 - |\ell_\alpha(f - f_0)| \ .$$

Now pick $\delta > 0$ so that whenever $\|f - f_0\| < \delta$,

$$|\ell_\alpha(f - f_0)| < \varepsilon/2 \ .$$

This proves the lower semi-continuity. By the previous theorem there exists a constant M such that $p(f) \leq M\|f\|$. □