1. The Theorem of Hille and Yosida concerning semi-groups

From now we consider X to be a Banach space.

Definition 1.1. *A family of bounded operators $P_t : X \to X, t \geq 0$ is a* **strongly continuous semi group** *if*
a) $P_0 = I$.
b) For any $s, t \geq 0$ we have that $P_{s+t} = P_s P_t$.
c) For any $f \in X$, $\lim_{t \to 0, t>0} \|P_t f - f\| = 0$.

Note that $t \to P_t f$ is continous for all $t \geq 0$, since

$$\lim_{\varepsilon \to 0} \|P_{t+\varepsilon} f - P_t f\| = \lim_{\varepsilon \to 0} \|P_\varepsilon P_t f - P_t f\| = 0 \ .$$

Such semi groups are natural in the context of linear evolution equations. An important sub-class are the contraction semi-groups.

Definition 1.2. *A family of bounded operators $P_t : X \to X, t \geq 0$ is a* **contraction semi-group** *if is a strongly continuous semi-group and for all $t \geq 0$*

$$\|P_t\| \leq 1 \ .$$

One would like to think of a semi group as an operator of the form e^{At} for some operator which one would call the generator of the semi-group. Let P_t be a contraction semi-group. Consider the set

$$D(A) = \{f \in X : \lim_{t \to 0} \frac{P_t f - f}{t} \text{ exists}\} \ .$$

On $D(A)$ define

$$Af = \lim_{t \to 0} \frac{P_t f - f}{t} \ .$$

Note that apriori $D(A)$ might consist only of the zero vector. We have, however, the following theorem.

Theorem 1.3. *The set $D(A)$ is dense in X and the operator A defined above is a linear closed operator. Further, if $f \in D(A)$, so is $P_t f$ for all $t \geq 0$ and $P_t A f = A P_t f$.*

Proof. Consider

$$V_t f = \frac{1}{t} \int_0^t P_s f ds$$

which exists as a Riemann integral, since the function $s \to P_s f$ is continous. V_t is a bounded operator since

$$\|V_t f\| \leq \frac{1}{t} \int_0^t \|P_s f\| ds \leq \|f\|$$

since P_t is a contraction. By the definition of the Riemann integral we also have that

$$\|V_t f - f\| \leq \frac{1}{t} \int_0^t \|P_s f - f\| ds$$

from which we see that

$$\lim_{t \to 0} \|V_t f - f\| = 0 \ .$$

In other words, the set

$$\cup_{t>0} V_t(X)$$

is dense in X. Notice that, because, P_t is bounded,

$$P_s V_t = V_t P_s \ . \tag{1}$$

For $t > 0$ w also find that

$$P_\varepsilon V_t f - V_t f = \frac{1}{t}\int_0^t P_{\varepsilon+s} f ds - \frac{1}{t}\int_0^t P_s f ds = \frac{1}{t}\int_\varepsilon^{t+\varepsilon} P_s f ds - \frac{1}{t}\int_0^t P_s f ds$$

$$= \frac{1}{t}\int_0^{t+\varepsilon} P_s f ds - \frac{1}{t}\int_0^t P_s f ds - \frac{1}{t}\int_0^\varepsilon P_s f ds = \frac{1}{t}\int_0^\varepsilon P_s P_t f ds - \frac{1}{t}\int_0^\varepsilon P_s f ds$$

so that

$$\frac{P_\varepsilon V_t f - V_t f}{\varepsilon} = \frac{1}{t}\left[V_\varepsilon P_t f - V_\varepsilon f\right] \tag{2}$$

which converges to $V_t f - f$ as $\varepsilon \to 0$. Hence $V_t f \in D(A)$ and

$$A V_t f = \frac{1}{t}\left[P_t f - f\right] \ .$$

This shows that $D(A) \in X$ is dense. Likewise, from (2) using (1),

$$V_t \frac{P_\varepsilon f - f}{\varepsilon} = \frac{1}{t}\left[P_t - I\right] V_\varepsilon f$$

and for $f \in D(A)$ we find that

$$V_t A f = \frac{1}{t}[P_t - I] f \tag{3}$$

and in particular,

$$A V_t f = V_t A f \ . \tag{4}$$

To see that A is closed, let $f_n \in D(A)$ be such that $f_n \to f$ and $A f_n \to v$. We have, using (3)

$$V_t v = \lim_{n\to\infty} V_t A f_n = \lim_{n\to\infty} \frac{1}{t}[P_t - I] f_n = \frac{1}{t}[P_t - I] f \ ,$$

since V_t is continuous. As $t \to 0$ the left side converges to V and hence the right side converges to which shows that $f \in D(A)$ and $Af = v$. Finally, for $f \in D(A)$

$$P_s \frac{1}{t}(P_t - I) f = \frac{1}{t}(P_t - I) P_s f$$

and the left side converges and hence so does the right side. Thus, $P_s f \in D(A)$ and

$$P_s A f = A P_s f \ .$$

□

We call A the **infinitesimal generator of** P_t.

Since P_t is a contraction, one can define the integral

$$R_\lambda(A) f := \int_0^\infty e^{-\lambda t} P_t f dt$$

for all $f \in X$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ as a Riemann integral.

Theorem 1.4. *The operator $R_\lambda(A)$ maps X into $D(A)$ and obeys the bound*

$$\|R_\lambda(A)\| \le \frac{1}{\mathrm{Re}\lambda} \ . \tag{5}$$

Moreover, for all $f \in X$

$$(\lambda I - A)R_\lambda(A)f = f \tag{6}$$

and for all $f \in D(A)$

$$R_\lambda(A)(\lambda I - A)f = f \ . \tag{7}$$

Thus, the resolvent set of A contains the right half plane and $R_\lambda(A) = (A - \lambda I)^{-1}$.

Proof. For $\mathrm{Re}\lambda > 0$ we find

$$\|R_\lambda(A)\| \le \int_0^\infty e^{-\mathrm{Re}\lambda t}\|P_t f\|dt \le \frac{1}{\mathrm{Re}\lambda}\|f\| \ .$$

Again we compute

$$[P_\varepsilon - I]R_\lambda(A)f = e^{\varepsilon\lambda}\int_\varepsilon^\infty e^{-\lambda t}P_t f dt - \int_0^\infty e^{-\lambda t}P_t f dt = (e^{\varepsilon\lambda} - 1)\int_\varepsilon^\infty e^{-\lambda t}P_t f dt - \int_0^\varepsilon e^{-\lambda t}P_t f dt$$

so that

$$\frac{P_\varepsilon - I}{\varepsilon}R_\lambda(A)f = \frac{(e^{\varepsilon\lambda} - 1)}{\varepsilon}\int_\varepsilon^\infty e^{-\lambda t}P_t f dt - \frac{1}{\varepsilon}\int_0^\varepsilon e^{-\lambda t}P_t f dt \ .$$

As $\varepsilon \to 0$ we see that the right side converges and hence so does the left side and

$$AR_\lambda(A)f = \lambda R_\lambda(A)f - f$$

which proves (6). To see (7) we assume that $f \in D(A)$ and write

$$R_\lambda(A)[P_\varepsilon - I]f = e^{\varepsilon\lambda}\int_\varepsilon^\infty e^{-\lambda t}P_t f dt - \int_0^\infty e^{-\lambda t}P_t f dt = (e^{\varepsilon\lambda} - 1)\int_\varepsilon^\infty e^{-\lambda t}P_t f dt - \int_0^\varepsilon e^{-\lambda t}P_t f dt$$

so that upon dividing by ε and taking the limit as $\varepsilon \to 0$ we get that

$$R_\lambda(A)Af = \lambda R_\lambda(A)f - f$$

which proves (7). The last statement follows from (5). □

Remark 1.5. *Note that we defined the resolvent to be $(\lambda I - A)^{-1}$ which differs from our usual definition by a minus sign.*

Lemma 1.6. *Let A be a closed densely defined operator and assume that $(0, \infty) \subset \rho(A)$ and that*

$$\|(\lambda I - A)^{-1}\| \le \frac{1}{\lambda}, \ \lambda > 0 \ .$$

Then

$$\lambda(\lambda I - A)^{-1}f \to f$$

as $\lambda \to \infty$.

Proof. Let $f \in D(A)$. Then

$$\lambda(\lambda I - A)^{-1}f - f = (\lambda I - A)^{-1}[\lambda I - \lambda I + A]f = (\lambda I - A)^{-1}Af$$

and therefore

$$\|\lambda(\lambda I - A)^{-1}f - f\| \le \frac{1}{\lambda}\|Af\| \ ,$$

which tends to 0 as $\lambda \to \infty$. If $f \in X$ for any $\varepsilon > 0$ we can find $g \in D(A)$ so that $\|f - g\| < \varepsilon$. Now

$$\lambda(\lambda I - A)^{-1} f - f = \lambda(\lambda I - A)^{-1}(f - g) - (f - g) + \lambda(\lambda I - A)^{-1} g - g \ .$$

The term

$$\lambda(\lambda I - A)^{-1}(f - g) - (f - g)$$

can be estimated

$$\|\lambda(\lambda I - A)^{-1}(f - g) - (f - g)\| \le \|\lambda(\lambda I - A)^{-1}(f - g)\| + \|f - g\| \le 2\|f - g\| = 2\varepsilon$$

and the second term tends to zero as $\lambda \to \infty$ which proves the lemma. □

Lemma 1.7. *With the same assumptions as in the previous lemma the operator*

$$\lambda(\lambda I - A)^{-1} A$$

is bounded and for any $f \in D(A)$

$$\|\lambda(\lambda I - A)^{-1} A f - A f\| \to 0$$

as $\lambda \to \infty$.

Proof.

$$\begin{aligned}\lambda(\lambda I - A)^{-1} A &= \lambda(\lambda I - A)^{-1}(A - \lambda I) + \lambda^2 (\lambda I - A)^{-1} \\ &= \lambda^2(\lambda I - A)^{-1} - \lambda\end{aligned}$$

and therefore for any $f \in D(A)$

$$\|\lambda(\lambda I - A)^{-1} A f\| = \|\lambda^2(\lambda I - A)^{-1} f - \lambda f\| \le 2\lambda \|f\| \ .$$

Since $D(A)$ is dense, this proves that $\lambda(\lambda I - A)^{-1} A$ is bounded. The other statement follows from the previous lemma. □

The idea is now to replace the operator A by the operator

$$A_\lambda := \lambda(\lambda I - A)^{-1} A$$

which is bounded. The semigroup

$$e^{A_\lambda t}$$

is now simply defined by the power series, which is norm convergent.

Lemma 1.8. *The operator*

$$e^{A_\lambda t} := \sum_{k=0}^{\infty} \frac{(A_\lambda t)^k}{k!}$$

is norm convergent and is a contraction semi group.

Proof. That it is norm convergent and a semi group is standard and the proof is left to the reader. Now,

$$\|e^{A_\lambda t}\| = \|e^{t(\lambda^2(\lambda I - A)^{-1} - \lambda)}\| \le e^{-\lambda t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \|\lambda^2(\lambda I - A)^{-1}\| \le e^{-\lambda t} \sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!} = 1$$

since

$$\|\lambda^2(\lambda I - A)^{-1}\| \le \lambda \ .$$

□

Theorem 1.9 (Hille-Yoshida). *A closed operator A is the generator of a contraction semi group if and only if*

$$(0,\infty) \subset \rho(A)$$

and

$$\|R_\lambda(A)\| \leq \frac{1}{\lambda},\ \lambda > 0\ .$$

Proof. We have to show that

$$\lim_{\lambda\to\infty} e^{A_\lambda t}$$

exists and defines a contraction semi group with infinitesimal generator A. Fix $\lambda > 0$ and $\mu > 0$ and write

$$e^{A_\lambda t} - e^{A_\mu t} = e^{A_\mu t + (A_\lambda - A_\mu)s}|_0^t = \int_0^t \frac{d}{ds} e^{A_\mu t + (A_\lambda - A_\mu)s} ds$$

which equals

$$\int_0^t e^{A_\mu(t-s)}(A_\lambda - A_\mu)e^{A_\lambda s} ds = \int_0^t e^{A_\mu(t-s)} e^{A_\lambda s}(A_\lambda - A_\mu) ds$$

where we have used that $A_\lambda A_\mu = A_\mu A_\lambda$ and that

$$e^{A_\mu t + (A_\lambda - A_\mu)s} = e^{A_\mu t} e^{(A_\lambda - A_\mu)s}\ .$$

Now for $f \in X$

$$\|[e^{A_\lambda t} - e^{A_\mu t}]f\| \leq \int_0^t \|e^{A_\mu(t-s)} e^{A_\lambda s}(A_\lambda - A_\mu)f\| ds$$

so that

$$\|[e^{A_\lambda t} - e^{A_\mu t}]f\| \leq t\|(A_\lambda - A_\mu)f\|\ . \tag{8}$$

If $f \in D(A)$ then

$$\|A_\lambda f - Af\| \to 0$$

as $\lambda \to \infty$ and hence $e^{A_\lambda t}f$ is a Cauchy sequence and hence converges. Since $D(A)$ is dense, by standard arguments, $e^{A_\lambda t}f$ converges to $P_t f$ for all $f \in X$ and the linear operator P_t is a contraction. We have to show that it is a semi group.

$$P_{t+s}f = \lim_{\lambda\to\infty} e^{A_\lambda(t+s)}f = \lim_{\lambda\to\infty} e^{A_\lambda t} e^{A_\lambda s} f$$

$$= \lim_{\lambda\to\infty} e^{A_\lambda t}[e^{A_\lambda s} - P_s]f + \lim_{\lambda\to\infty} e^{A_\lambda t} P_s f$$

Now note that

$$\|e^{A_\lambda t}[e^{A_\lambda s} - P_s]f\| \leq \|[e^{A_\lambda s} - P_s]f\|$$

which tends to zero as $\lambda \to \infty$ and

$$\lim_{\lambda\to\infty} e^{A_\lambda t} P_s f = P_t P_s f\ .$$

Since

$$\|[P_t - I]f\| \leq 2\|f\|$$

it suffices to show that

$$\lim_{t\to 0} \|[P_t - I]f\| = 0$$

for a dense set of vectors f. Pick $f \in D(A)$. Then by (8) we have that

$$\|[P_t - e^{A_\mu t}]f\| \leq t\|(A - A_\mu)f\| \tag{9}$$

and hence

$$\|[P_t - I]f\| \le \|[P_t - e^{A_\mu t}]f\| + \|[e^{A_\mu t} - I]f\| \le t\|(A - A_\mu)f\| + \|[e^{A_\mu t} - I]f\| \to 0$$

as $t \to 0$. Thus, we have shown that P_t is a contraction semi group and therefore it has a generator B. We have shown that necessarily $\rho(B)$ contains the complex numbers with positive real part and, moreover,

$$\|(\lambda I - B)^{-1}\| \le \frac{1}{\mathrm{Re}\lambda} \ .$$

For $f \in X$ we find

$$e^{A_\lambda t} f - f = \int_0^t e^{A_\lambda s} A_\lambda f ds$$

and find that for $f \in D(A)$

$$P_t f - f = \int_0^t P_s A f ds \ .$$

From this we find that for $f \in D(A)$

$$\lim_{t \to 0} \frac{[P_t - I]f}{t} = Af$$

and hence $A \subset B$. But for arbitrary $g \in X$

$$(\lambda I - B)(\lambda I - A)^{-1} g = (\lambda I - A)(\lambda I - A)^{-1} g = g$$

and hence

$$(\lambda I - B)(\lambda I - A)^{-1} = I$$

so

$$(\lambda I - A)^{-1} = (\lambda I - B)^{-1}$$

and therefore $D(B) = D(A)$. □