1. Basic theorem on self adjointness

The following theorem is basic to the theory of self adjoint operators. It clarifies the role played by the adjoint of a symmetric operator.

Theorem 1.1. *Let A be a symmetric operator on a Hilbert space $\mathcal{H}$, i.e., A is densely defined and for all $f, g \in D(A)$*

$$\langle Af, g\rangle = \langle f, Ag\rangle \ .$$

Then the following three statements are equivalent, i.e., each of them implies the other two.

a) A is self adjoint,

b) A is closed and $\mathrm{Ker}(A^* \pm iI) = \{0\}$,

c) $\mathrm{Ran}(A \pm iI) = \mathcal{H}$.

Proof. We assume that $A = A^*$ and prove b). Since A^* is closed so is A. Since a self adjoint operator has only realy eigenvalues, $\mathrm{Ker}(A^* \pm iI) = \{0\}$. Next we assume b) and prove c). The range of $(A + iI)$ is dense, for if $f \perp \mathrm{Ran}(A + iI)$ then

$$\langle (A + iI)g, f\rangle = 0$$

for all $g \in D(A)$ and hence

$$\langle Ag, f\rangle = -i\langle g, f\rangle \ .$$

This implies that $f \in D(A^*)$ and therefore

$$0 = \langle g, (A^* - iI)f\rangle$$

for all $g \in D(A)$. Since $D(A)$ is dense, it follows that $f \in \mathrm{Ker}(A^* - iI)$ and hence $f = 0$. The argument is the same for $\mathrm{Ran}(A - iI)$. Next we show that $\mathrm{Ran}(A \pm iI)$ is closed. For any $f \in D(A)$ we have

$$\|(A + iI)f\|^2 = \|Af\|^2 + \|f\|^2$$

since A is symmetric. Thus,

$$\|(A + iI)f\|^2 \geq \|f\|^2 \ . \tag{1}$$

If $g_n \in \mathrm{Ran}(A + iI)$ is a sequence that converges to g in $\mathcal{H}$ then $g_n = (A + iI)f_n$ for some $f_n \in D(A)$. The inequality (1) now implies that f_n is a Cauchy sequence and hence converges to some element f. Since A is closed we must have that $f \in D(A)$ and $(A+iI)f = g$ and hence $g \in \mathrm{Ran}(A + iI)$. Thus we conclude that $\mathrm{Ran}(A + iI) = \mathcal{H}$. The proof for $\mathrm{Ran}(A - iI) = \mathcal{H}$ is the same. Next, we prove that c) implies a). Since A is symmetric, $A \subset A^*$. It remains to show that $D(A^*) \subset D(A)$. Let $g \in D(A^*)$. Since $\mathrm{Ran}(A + iI) = \mathcal{H}$ there exists $h \in D(A)$ with

$$(A^* + iI)g = (A + iI)h$$

or

$$A^*(g - h) = -i(g - h)$$

since $h \in D(A^*)$. Thus, $g - h \in \mathrm{Ker}(A^* + iI)$. Since $\mathrm{Ran}(A - iI) = \mathcal{H}$, $\mathrm{Ker}(A^* + iI) = \{0\}$ and hence $g = h$. Just note that for $f \in \mathrm{Ker}(A^* + iI)$ we have for all $g \in D(A)$

$$0 = \langle g, (A^* + iI)f\rangle = \langle (A - iI)g, f\rangle$$

which implies that $f = 0$ since $\mathrm{Ran}(A - iI) = \mathcal{H}$. □

At first sight it is hard to imagine that the adjoint of a symmetric operator can have an imaginary eigenvalue. Here is an example due to von Neumann. Consider the operator

$$A = \frac{1}{i}\frac{d}{dx}x^3 + x^3\frac{1}{i}\frac{d}{dx}$$

on the domain $D(A) = C_c^\infty(\mathbb{R})$. To be precise for $f \in D(A)$

$$Af(x) = \frac{1}{i}\frac{d}{dx}(x^3 f)(x) + \frac{1}{i}x^3 f'(x)$$

The operator A is symmetric. This is a simple exercise. Consider now the equation

$$\frac{1}{i}\frac{d}{dx}(x^3 f) + x^3\frac{1}{i}\frac{df}{dx} = if \ .$$

Note that f in this equation is not in $D(A)$. So the computation is a formal one. This equation is the same as

$$3x^2 f(x) + 2x^3 f'(x) = -f(x) \ ,$$

a first order linear equation which can be solved explicitly.

$$f'(x) = -(\frac{3}{2x} + \frac{1}{2x^3})f(x)$$

or

$$f(x) = \text{const.}|x|^{-3/2}e^{-\frac{1}{4x^2}} \ .$$

If we set $f(0) = 0$ for $x = 0$, the function is everywhere defined and differentiable, in fact infinitely often differentiable. The function f is in $L^2(\mathbb{R})$ and hence $f \in D(A^*)$. So we have found $f \neq 0, f \in L^2(\mathbb{R})$ such that

$$A^* f = if \ .$$

Recall that

$$\langle Ag, g\rangle = \langle g, Ag\rangle$$

for all $g \in D(A)$. To understand this a bit better for the case at hand, consider

$$\int_{-R}^{R}\left[\frac{1}{i}\frac{d}{dx}(x^3 f) + x^3\frac{1}{i}\frac{df}{dx}\right]\overline{f}dx$$

which, using integration by parts, equals

$$2\frac{1}{i}x^3|f(x)|^2\Big|_{-R}^{R} + \int_{-R}^{R} f\overline{\left[\frac{1}{i}\frac{d}{dx}(x^3 f) + x^3\frac{1}{i}\frac{df}{dx}\right]}dx \ .$$

Here R is positive. For our function f we see that

$$2\frac{1}{i}x^3|f(x)|^2\Big|_{-R}^{R} = \text{const.}^2 4\frac{1}{i}e^{-\frac{1}{2R^2}}$$

which does not converge to zero as $R \to \infty$.

Definition 1.2. *A one parameter unitary group $t \to U_t$ is defined by the following properties: For each $t \in \mathbb{R}$, U_t is a unitary operator, i.e., isometric and invertible. Moreover,*

$$U_0 = I \text{ and } U_{t+s} = U_t U_s \text{ for all } t, s \in \mathbb{R} \ .$$

For $f \in \mathcal{H}$

$$\lim_{t\to 0}\|U_t f - f\| = 0 \ .$$

Note that

$$U_t^{-1} = U_t^* = U_{-t} \ .$$

Theorem 1.3. *A closed, densely defined operator B is the generator of a unitary group if and only if $B = iA$, where $A = A^*$, i.e., A is self adjoint.*

Proof. Given a one parameter unitary group U_t we define the generator B as we did for semi groups. We know that B is closed and densely defined. The only property to show is self adjointness, or more properly skew adjointness of B. Let $f \in D(B)$. Then for all $v \in D(B)$ we have that

$$(f, Bv) = \lim_{t\to 0}(f, \frac{1}{t}(U_t - I)v) = \lim_{t\to 0}(\frac{1}{t}(U_{-t} - I)f, v)$$

and

$$\lim_{t\to 0}(\frac{1}{t}(U_{-t} - I)f = \lim_{t\to 0}(\frac{1}{t}U_{-t}(I - U_t)f = -Bf$$

since U_t is continuous. Hence for $f \in D(B)$ we have for all $v \in D(B)$

$$(f, Bv) = (-Bf, v)$$

and hence B is skew symmetric, i.e., $B = iA$ where A is symmetric, $A \subset A^*$. As in the proof of the Hille-Yoshida theorem we define

$$R_\lambda(\pm iA)f = \int_0^\infty e^{-\lambda t} U_{\pm t} f dt \ , \Re\lambda > 0$$

which exists as a Riemann integral and find that

$$(\lambda I \pm iA)R_\lambda(\pm iA) = I$$

and

$$R_\lambda(\pm iA)(\lambda I \pm iA)f = f$$

for all $f \in D(B)$. In particular $\mathrm{Ran}(\lambda I \pm iA) = \mathcal{H}$. Hence A is self adjoint. Conversely, we assume that $A = A^*$. Consider first the operator $B = iA$. Because A is self adjoint we know that

$$\mathrm{Ran}(\lambda I - B) = \mathrm{Ran}(-i\lambda I - A) = \mathcal{H}$$

for all $\lambda \in \mathbb{R}, \lambda \neq 0$. Moreover, for all $f \in D(B)$,

$$\|(\lambda I - B)f\|^2 = \|(-i\lambda - A)f\|^2 = \lambda^2\|f\|^2 + \|Af\|^2$$

and hence the resolvent $(\lambda I - B)^{-1}$ exists on $\mathcal{H}$ with the bound

$$\|(\lambda I - B)^{-1}\| \leq \frac{1}{\lambda}$$

for $\lambda > 0$. The B is the generator of a contraction semigroup, which we denote by V_t. For $f \in D(B)$ we compute noting that $V_t f \in D(B)$,

$$\frac{d}{dt}\|V_t f\|^2 = (BV_t f, V_t f) + (V_t f, Bv_t f) = (iAV_t f, V_t f) + (V_t f, iAv_t f) = (V_t f, [-iA + iA]V_t f) = 0$$

since A is self adjoint. Hence V_t is an isometry. We may apply the same reasoning to the operator $-B$ and obtain an isometric semi group W_t. Next for $f \in D(B) = D(-B)$ we compute

$$\frac{d}{dt}W_t V_t f = W_t(-B)V_t f + W_t B V_t f = 0$$

and thus $W_t V_t = I$. Note that we have that $W_t f \in D(B)$ as well as $V_t f \in D(B)$. The same reasoning shows that $V_t W_t = I$. Hence V_t is a unitary group where we set $V_{-t} = W_t$.

$\square$