1. Extensions of symmetric operators

References: Joachim Weidmann, Linear Operators in Hilbert Spaces, Springer-Verlag,

In what follows $\mathcal{H}$ is a **complex** Hilbert space.

Recall that an operator $A : D(A) \to H$ is symmetric if $D(A)$ is dense in $\mathcal{H}$ and for all $f, g \in D(A)$

$$(f, Ag) = (Af, g) \ .$$

Since $A \subset A^*$ it follows that a symmetric operator is closable and we shall henceforth assume that A is a closed symmetric operator. Note that this assumption entails that the spaces $\mathrm{Ran}(A + iI)$ and $\mathrm{Ran}(A - iI)$ are subspaces of $\mathcal{H}$, i.e., closed linear manifolds.

We define the Cayley transform of A,

$$V : \mathrm{Ran}(A + iI) \to \mathrm{Ran}(A - iI)$$

which is defined as follows. For $g \in \mathrm{Ran}(A + iI)$ there exists a unique $f \in D(A)$ with

$$g = (A + iI)f \ .$$

Define

$$Vg = (A - iI)f \ .$$

The operator V satisfies $\|Vg\| = \|g\|$ and V is onto. We call such operators **isometries**. The point of the next theorem is that the study of symmetric operators can be reduced to the study of isometries.

Theorem 1.1. (One to one correspondence between symmetric operators and isometries) *Let A be a closed symmetric operator and V its Cayley transform. Then $\mathrm{Ran}(I - V)$ is dense in $\mathcal{H}$. Conversely let F and G be two subspaces of a Hilbert space and assume that $V : F \to G$ is an isometry such that $\mathrm{Ran}(I - V)$ is dense in $\mathcal{H}$. Then V is the Cayley transform of a closed symmetric operator.*

Proof. For any $g \in \mathrm{Ran}(A + iI)$ there exists a unique $f \in D(A)$ such that $g = (A + iI)f$ and $Vg = (A - iI)g$. Hence

$$f = \frac{1}{2i}(g - Vg) \text{ and } Af = \frac{1}{2}(g + Vg) \ ,$$

$$D(A) = \mathrm{Ran}(\frac{1}{2i}(I - V)$$

and since $D(A)$ is dense in $\mathcal{H}$ the first statement follows. To see the converse define $D(A) = \mathrm{Ran}(I - V)$. I.e., for $f \in D(A)$ there exists $g \in F$ such that

$$f = \frac{1}{2i}(g - Vg) \ .$$

By assumption this set is dense in $\mathcal{H}$. In fact the vector g is unique. This is equivalent to the statement that $(I - V)$ is injective. If there exists $h \in F$ with $Vh = h$ then we have for all $f \in F$

$$(h, (I - V)f) = (h, f) - (h, Vf) = (h, f) - (Vh, Vf) = (h, f) - (h, f) = 0$$

since V preserves the inner product. Since $\mathrm{Ran}(I-V)$ is dense it follows that $h=0$. Thus for any $f \in D(A)$ there exists a unique $g \in F$ such that $f = \frac{1}{2i}(g - Vg)$. Next we define

$$Af = \frac{1}{2}(g + Vg)$$

and note that for any $f_1, f_2 \in D(A)$

$$(f_1, Af_2) = -\frac{1}{4i}((g_1 - Vg_1),(g_2 + Vg_2)) = -\frac{1}{4i}\left[(g_1,g_2) + (g_1,Vg_2) - (Vg_1,g_2) - (g_1,g_2)\right] =$$

$$-\frac{1}{4i}\left[(g_1,Vg_2) - (Vg_1,g_2)\right] \ .$$

Likewise,

$$(Af_1, f_2) = \frac{1}{4i}((g_1 + Vg_1),(g_2 - Vg_2)) = \frac{1}{4i}\left[-(g_1,Vg_2) + (Vg_1,g_2)\right]$$

and hence A is symmetric. To see that A is closed let $g_n \in D(A)$ be a sequence converging to g and Ag_n converging to h. Then the sequence

$$f_n = (A + iI)g_n$$

converges to $f := h + ig$ which, since F is closed, is also in F. Likewise, $Vf_n = (A - iI)g_n$ converges to $h - ig$ and since G is closed we have that $h - ig \in G$ and $Vf = h - ig$. Hence we have that

$$g = \frac{1}{2i}(f - Vf) \text{ and } h = \frac{1}{2}(f + Vf) \ ,$$

which means that $g \in D(A)$ and $h = Ag$. □

A simple consequence is the

Corollary 1.2. *The Cayley transform of a closed symmetric operator A is unitary. Conversely, a unitary operator V such that* $\mathrm{Ran}(I-V)$ *is dense is the Cayley transform of a self adjoint operator.*

Proof. Suppose that A is self adjoint. Then by the basic theorem on self adjoint operators $\mathrm{Ran}(A \pm iI) = \mathcal{H}$. Thus the isometry $V : \mathcal{H} \to \mathcal{H}$ is onto, injective and hence unitary. Conversely, suppose that V is unitary and $\mathrm{Ran}(I-V)$ dense, in particular the space $F = \mathcal{H}$. Then the operator A is defined on $\mathrm{Ran}(I-V)$ by

$$Af = \frac{1}{2}(g + Vg)$$

where g is the unique solution of the equation

$$f = \frac{1}{2i}(g - Vg) \ .$$

We note that

$$(A + iI)f = g \ , \ (A - iI)f = Vg$$

and since g can be chosen arbitrarily in $\mathcal{H}$ and since V is unitary, this shows that $\mathrm{Ran}(A \pm iI) = \mathcal{H}$ and hence $A = A^*$.

□

Our next theorem pushes this correspondence further by showing that extensions of symmetric operators correspond to extensions of isometries.

Theorem 1.3. (**Extensions of symmetric operators correspond to extensions of isometries**) *The operator A' is a closed symmetric extension of a closed symmetric operator A if and only if for the corresponding Cayley transforms $V \subset V'$.*

Proof. Let A' be a symmetric closed extension. Then $\mathrm{Ran}(A+iI) \subset \mathrm{Ran}(A'+iI)$ and $\mathrm{Ran}(A-iI) \subset \mathrm{Ran}(A'-iI)$. For $g \in \mathrm{Ran}(A+iI)$ there exists a unique $f \in D(A)$ such that $g = (A+iI)f$ and $Vg = (A-iI)f$. Since $A \subset A'$, $g = (A'+iI)f$ and $V'g = (A'-iI)f = Vg$. Hence $V \subset V'$. Conversely if $V \subset V'$ then for $g \in D(A)$ we have $g = \frac{1}{2i}(f - Vf)$ for a unique $f \in \mathrm{Ran}(A+iI)$. Since $V \subset V'$ we have that $\mathrm{Ran}(A+iI) \subset \mathrm{Ran}(A'+iI)$ and $\mathrm{Ran}(A-iI) \subset \mathrm{Ran}(A'-iI)$. Finally, $Ag = \frac{1}{2}(f+Vf) = \frac{1}{2}(f+V'f) = A'g$. Hence $A \subset A'$. $\square$

The next step in our program consists of understanding extensions of isometries.

Theorem 1.4. (**Structure of isometric extensions**) *Let F, G be two subspaces of $\mathcal{H}$ and $V : F \to G$ an isometry, i.e., $\|Vf\| = \|f\|$ for all $f \in F$ and V is onto G. Let $F_+ \subset F^\perp$ and $F_- \subset G^\perp$ be subspaces. Then*

$$V' : F \oplus F_+ \to G \oplus F_-$$

is an isometric extension of V if and only if $\dim F_+ = \dim F_-$ and there exists an isometry $\widetilde{V} : F_+ \to F_-$ so that for any $f \in F \oplus F_+$, i.e., $f = f_0 + f_+, f_0 \in F, f_+ \in F_+$

$$V'f = Vf_0 + \widetilde{V} f_- \ . \tag{1}$$

Proof. Clearly V' of the form given above is an isometry. It is onto $G \oplus F_-$ since V is onto G and $\widetilde{V}$ is onto F_-. It preserves length since

$$(V'f, V'f) = (Vf_0 + \widetilde{V}f_+, Vf_0 + \widetilde{V}f_+) = (Vf_0, Vf_0) + (Vf_0, \widetilde{V}f_+) + (\widetilde{V}f_+, Vf_0) + (\widetilde{V}f_+, \widetilde{V}f_+) \ .$$

Since $Vf_0 \in G$ and $\widetilde{V}f_+ \in F_-$ they are perpendicular to each other and hence

$$(V'f, V'f) = (Vf_0 + \widetilde{V}f_+, Vf_0 + \widetilde{V}f_+) = (Vf_0, Vf_0) + (\widetilde{V}f_+, \widetilde{V}f_+)$$

$$= (f_0, f_0) + (f_+, f_+) = (f, f) \ .$$

Suppose that $V' : F \oplus F_+ \to G \oplus F_-$ is an isometry that extends V. For any vector $f_+ \in F_+$ define

$$\widetilde{V}f_+ = V'f_+ \in G \oplus F_- \ .$$

Pick any $g \in G$ and note that since V is onto G there exists $f \in F$ with $g = Vf$. Hence

$$(g, \widetilde{V}f_+) = (Vf, V'f_+) = (V'f, V'f_+) = (f, f_+) = 0$$

Since $g \in G$ is arbitrary, $\widetilde{V}f_+ \perp G$ and hence $\widetilde{V}$ maps F_+ into F_-. That $\widetilde{V}$ preserves the norm follows from the fact that V' does. We have to show that $\widetilde{V}$ is onto. For any $f_- \in F_-$ there exists $f \in F \oplus F_+$ so that $V'f = f_-$ because V' is onto. For any $f_0 \in F$ we have that

$$0 = (Vf_0, V'f) = (V'f_0, V'f) = (f_0, f)$$

Hence $f \in F_+$ and $\widetilde{V}f = f_-$. The formula (1) is obvious. $\square$

With these results we can now completely characterize all closed symmetric extensions of a closed symmetric operator.

Theorem 1.5. (Closed symmetric extensions) *Let $A : D(A) \to \mathcal{H}$ be a closed symmetric operator. A closed symmetric operator $A' : D(A') \to \mathcal{H}$ is an extension of A if and only if the following holds:*
There exists subspaces $F_+ \subset \mathrm{Ker}(A^* - iI) = \mathrm{Ran}(A + iI)^\perp, F_- \subset \mathrm{Ker}(A^* + iI) = \mathrm{Ran}(A - iI)^\perp$ and an isometry $\widetilde{V} : F_+ \to F_-$ so that the Cayley transform of A', V' is of the form

$$V' : \mathrm{Ran}(A + iI) \oplus F_+ \to \mathrm{Ran}(A - iI) \oplus F_-$$

$$V'f = V f_0 + \widetilde{V} f_+ \tag{2}$$

with $f = f_0 + f_+$.
In particular this entails that $\mathrm{dim} F_+ = \mathrm{dim} F_-$.

Proof. If A' is a closed symmetric extension of A then we know that for their respective Cayley transforms $V \subset V'$ by Theorem 1.3. Since $\mathrm{Ran}(A \pm iI) \subset \mathrm{Ran}(A' \pm iI)$ we can define $F_\pm$ to be the orthogonal complement of $\mathrm{Ran}(A \pm iI)$ in $\mathrm{Ran}(A' \pm iI)$. The conclusion follows from Theorem 1.4. Conversely, if V' is given by (2) then $\mathrm{Ran}(I - V')$ is dens in $\mathcal{H}$ because $\mathrm{Ran}(I - V)$ is and hence by Theorem refsymmetric isometries V' is the Cayley transform of a closed symmetric operator and since $V \subset V'$ we have that A' extends A. □

Corollary 1.6. *A closed symmetric operator has self-adjoint extensions if and only if the deficiency indices*

$$n_\pm := \dim \mathrm{Ker}(A^* \mp iI)$$

are equal.

Proof. This follows from the fact that a closed symmetric operator is self-adjoint if and only if its Cayley transform is unitary. □

A useful fact about deficiency indices is the following theorem.

Theorem 1.7. *For any $\mu > 0$ the function*

$$n_\pm(\mu) := \dim \mathrm{Ker}(A^* \mp i\mu I)$$

is constant.

Proof. Consider two closed subspaces $F, G \subset \mathcal{H}$ and consider the orthogonal projections P_F, P_G onto F resp. G. We claim that if the norm $\|P_F - P_G\| < 1$, then the the two spaces have the same dimension. If the dimensions are both infinity, there is nothing to prove. So suppose that

$$\mathrm{dim} F > \mathrm{dim} G \ .$$

There exists a non-zero vector $f \in F$ with $f \perp G$. (Why?) Hence

$$(P_F - P_G) f = f$$

which contradicts the assumption that $\|P_F - P_G\| < 1$. Hence, $\mathrm{dim} F \le \mathrm{dim} G$. The result follows by exchanging the roles of F and G. Consider the orthogonal projections P, P' that project the Hilbert space onto $\mathrm{Ker}(A^* - i\mu I)$ and $\mathrm{Ker}(A^* - i\mu' I)$. We note that for any $f \in D(A)$ we have that

$$\|(A + i\mu I) f\| \ge |\mu| \|f\| \ .$$

For any $h \in H$ we have that

$$\|(I - P)h\| = \sup_{f \in D(A)} \frac{|(h, (A + i\mu I) f)|}{\|(A + i\mu I) f\|}$$

noting that the right side is the projection of h onto $\text{Ran}(A+i\mu)$. Likewise,

$$\|(I-P')h\| = \sup_{f\in D(A)} \frac{|(h,(A+i\mu' I)f)|}{\|(A+i\mu' I)f\|} \ .$$

If $h \in \text{Ker}(A^* - i\mu I)$, then

$$\|(I-P')h\| = \sup_{f\in D(A)} \frac{|(h,(A+i\mu' I)f)|}{\|(A+i\mu' I)f\|} = \sup_{f\in D(A)} \frac{|((A^*-i\mu' I)h,f)|}{\|(A+i\mu' I)f\|} = \sup_{f\in D(A)} \frac{|\mu-\mu'||(h,f)|}{\|(A+i\mu' I)f\|}$$

and the last term can is bounded by

$$\frac{|\mu-\mu'|}{|\mu'|}\|h\| \ .$$

Thus, we have shown that for any h with $Ph = h$,

$$\|(P-P')h\| = \|(I-P')h\| \le \frac{|\mu-\mu'|}{|\mu'|}\|h\| \ .$$

Thus if $\frac{|\mu-\mu'|}{|\mu'|} < 1$ we have that $\|P-P'\| < 1$ and the dimensions are the same. □

So far this has been rather abstract. More useful are the following two theorems of von Neumann. The first one is simple but instructive since it allows, in principle, to compute a special extension of A, namely A^*.

Theorem 1.8. *Let $A : D(a) \to \mathcal{H}$ be a closed symmetric operator. Then*

$$D(A^*) = D(A) + \text{Ker}(A^* - iI) + \text{Ker}(A^* + iI)$$

where the sum is direct. Moreover, writing an arbitrary $f \in D(A^*)$ as $f = f_0 + f_+ + f_-$ where $f_0 \in D(A)$, $f_+ \in \text{Ker}(A - iI)$ and $f_- \in \text{Ker}(A + iI)$ we have

$$A^* f = Af_0 + if_+ - if_- \ .$$

Proof. Since $A \subset A^*$ we have that $D(A) \subset D(A*)$. The inclusions $\text{Ker}(A - iI) \subset D(A^*)$ and $\text{Ker}(A^* + iI) \subset D(A^*)$ are obvious. Hence

$$D(A) + \text{Ker}(A^* - iI) + \text{Ker}(A^* + iI) \subset D(A^*).$$

To see the converse, consider any $f \in D(A^*)$. Since the subspaces $\text{Ran}(A + iI)$ and $\text{Ran}(A + iI)^\perp = \text{Ker}(A^* - iI)$ are closed we can split the vector $(A^* + iI)f$ uniquely into $u + f_+$ where $u = \text{Ran}(A+iI)f_0$ for some unique $f_0 \in D(A)$ and $A^* f_+ = if_+$ or $(A^*+iI)vf_+ = 2if_+$. Thus,

$$(A^* + iI)f = (A + iI)f_0 + (A^* + iI)[\frac{1}{2i}f_+ = (A^* + iI)f_0 + (A^* + iI)[\frac{1}{2i}f_+$$

since $D(A) \subset D(A^*)$. Hence

$$(A^* + iI)\left[f - f_0 - \frac{1}{2i}f_+\right] = 0 \ .$$

This means that $f - f_0 - \frac{1}{2i}f_+ \in \text{Ker}(A^* + iI)$ and hence

$$D(A^*) = D(A) + \text{Ker}(A^* - iI) + \text{Ker}(A^* + iI) \ .$$

To see that the sum is direct, assume that $f_0 + f_+ + f_- = 0$ where $f_0 \in D(A)$, $f_+ \in \text{Ker}(A^* - iI)$ and $f_- \in \text{Ker}(A^* + iI)$. Now

$$(A + iI)f_0 = (A^* + iI)f_0 = (A^* + iI)f_+ = 2if_+$$

This says that $f_+ \in \text{Ker}(A^* - iI) \cap \text{Ran}(A + iI)$ which are orthogonal complements. Hence $f_+ = 0$. In a similar fashion we see that $f_- = 0$ and hence $f_0 = 0$ and the sum is direct. Finally, for $f \in D(A^*)$, i.e., $f = f_0 + f_+ + f_-$ we compute

$$A^* f = Af_0 + A^* f_+ + A^* f_- = Af_0 + if_+ - if_- \ .$$

□

Now we extend this theorem to arbitrary closed symmetric extensions. This time, we will make use of the Cayley transform.

Theorem 1.9. *Let A be a closed symmetric operator. The operator A' is a closed symmetric extension of A if and only if there are subspaces $F_\pm \subset \text{Ker}(A^* \mp iI)$ and an isometry $\widetilde{V}$: $F_+ \to F_-$ such that*

$$D(A') = D(A) + (I - \widetilde{V})(F_+) \tag{3}$$

where the sum is direct and for any $f \in D(A')$ we have $f = f_0 + g - \tilde{V}g$ for a unique $f_0 \in D(A)$ and $g \in \mathcal{H}$. Further,

$$A'(f_0 + g - \widetilde{V}g) = Af_0 + ig + i\widetilde{V}g \ . \tag{4}$$

The operator A' is self adjoint if and only if $F_\pm = \text{Ker}(A^* \mp iI)$.

Proof. Using Theorem 1.4 it remains to show the displayed formulas. The domain of A' is given by $\text{Ran}(I - V') = (I - V)(\text{Ran}(A + iI) + \text{Ran}(I - \widetilde{V})(F_+)$. Since $(I - V)(\text{Ran}(A + iI) = D(A)$ the formula (3) is established. Since $F_\pm \subset \text{Ker}(A^* \mp iI)$ it follows from the proof of the previous theorem that the sum is direct. Let $f \in D(A')$. There exists a unique $h \in \text{Ran}(A' + iI) = \text{Ran}(A + iI) \oplus F_+$ such that

$$f = (h - V'h)$$

and A' is then given by

$$A'f = i(h + V'h) \ .$$

Since h can be written as $h_0 + f_+$, $h_0 \in \text{Ran}(A + iI)$, $f_+ \in F_+$ and $V'(h_0 + f_+) = Vh_0 + \widetilde{V}f_+$ we have

$$f = (h_0 - Vh_0) + (f_+ - \widetilde{V}f_+) = f_0 + (f_+ - \widetilde{V}f_+)$$

and hence

$$A'(f_0 + f_+ - \widetilde{V}f_+) = Af_0 + i(f_+ + \widetilde{V}f_+) \ .$$

□

Example:

We apply now this theory to a concrete problem (taken from Reed -Simon, Modern methods of mathematical physics, volume I). Consider the Hilbert space $L^(0,1)$ of complex valued square integrable functions. Consider the dense domain

$$D = \{f \in L^2(0,1) : f \in AC[0,1], f(0) = f(1) = 0\} \ .$$

Here AC[0,1] is the space of absolute continuous functions whose derivative is square integrable. On D consider the operator

$$Af = \frac{1}{i}f' \ .$$

We have seen before that A is closed, symmetric but not self adjoint. We shall determine all its self adjoint extensions. it was shown in previous lectures that the adjoint of A, A^* is given by

$$A^* f = \frac{1}{i} f'$$

for f in the domain AC[01]. Note, there are no additional boundary conditions. That $f_+ \in \mathrm{Ker}(A^* - iI)$ means that $\frac{1}{i} f_+' = i f_+$, i.e., $f_+' = -f_+$. Hence

$$f_+(x) = \frac{\sqrt{2}}{\sqrt{e^2-1}} e^{-(x-1)} \ .$$

Note that the pre-factor renders the function normalized in $L^2(0,1)$. Likewise to find $f_- \in \mathrm{Ker}(A^* + iI)$ amounts to solving the equation $f' = f$ and we get

$$f_-(x) = \frac{\sqrt{2}}{\sqrt{e^2-1}} e^{x} \ .$$

We have tacitly assumed that every function that is in AC[01] and satisfies the equation $A^* f = f$ must be a multiple of the exponential function. This is certainly true for a function that is everywhere differentiable. Proof: Let f be any solution and write $f = g e^x$. Then $g e^x = f = f' = g' e^x + g e^x$ and hence $g' = 0$. We know from calculus that a function that is differentiable and whose derivative vanishes everywhere must be a constant function. The problem is that we know apriori only that the solution is in AC[01] and not necessarily differentiable everywhere. One approach on how to deal with this problem is the following: consider $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$ where $\phi(x) \in C_c^\infty(-1,1)$ and write $f_\varepsilon(x) := \int_0^1 \phi_\varepsilon(x-y) f(y) dy$. For all $x \in (\varepsilon, 1-\varepsilon)$ we have that f_ε is infinitely differentiable and its derivative is given by

$$f_\varepsilon'(x) = A^* f_\varepsilon(x) = \int_0^1 \phi_\varepsilon'(x-y) f(y) dy = -(A\phi_\varepsilon'(x-\cdot), f) = (\phi_\varepsilon(x-\cdot), A^* f)$$

noting that for $\varepsilon < x < 1-\varepsilon$ there are no boundary terms in the integration by parts. Since $A^* f = f$ we have that $f_\varepsilon'(x) = f_\varepsilon(x)$ and hence $f_\varepsilon(x) = c_\varepsilon e^x$. Since f is continuous, we have that $\lim_{\varepsilon \to 0} f_\varepsilon(x) = f(x)$ and hence $f(x) = c e^x$ where c is a constant. Thus, there are no other solutions and hence the operator A has deficiency indices $(1,1)$ and therefore has self-adjoint extensions.

Now, we use Theorem 1.9. Any isometry $\widetilde{V} : \mathrm{Ker}(A^* - iI) \to \mathrm{Ker}(A^* + iI)$ is given by

$$\widetilde{V}_\beta f_+ = \beta f_-$$

where β is a complex number of absolute value 1. Hence the domain of A_β, the self-adjoint extension corresponding to this isometry $\widetilde{V}_\beta$, is given by functions of the form

$$f_0 + c(f_+ - \beta f_-) = f_0 + c\frac{\sqrt{2}}{\sqrt{e^2-1}}(e^{-(x-1)} - \beta e^x)$$

where $f_0 \in D$ and c is an arbitrary complex constant. Further,

$$A_\beta(f_0 + c(f_+ - \beta f_-)) = Af_0 + ic(f_+ + \beta f_-) = Af_0 + ic\frac{\sqrt{2}}{\sqrt{e^2-1}}(e^{-(x-1)} + \beta e^x) \ .$$

The domain $D(A_\beta)$ has another characterization. Any function of the form $f := f_0 + c\frac{\sqrt{2}}{\sqrt{e^2-1}}(e^{-(x-1)} - \beta e^x)$ where $f_0 \in D$ satisfies

$$f(0) = c\frac{\sqrt{2}}{\sqrt{e^2-1}}(e-\beta) \text{ and } f(1) = c\frac{\sqrt{2}}{\sqrt{e^2-1}}(1-\beta e) \ .$$

In other words

$$f(1) = \frac{1-\beta e}{e-\beta} f(0) \ .$$

Note that the constant

$$\alpha := \frac{1-\beta e}{e-\beta}$$

is a complex constant of absolute value 1. Also note that

$$\beta = \frac{\alpha e - 1}{\alpha - e} \ .$$

Now let's turn things around, and consider the operator $B_\alpha f = \frac{1}{i} f'$ on the domain

$$D(B_\alpha) = \{f \in L^2(0,1) : f \in AC[0,1], f(1) = \alpha f(0)\} \ .$$

One expects that $A_\beta = B_\alpha$. To see this we have to show that the domains are equal. We have seen that $D(A_\beta) \subset D(B_\alpha)$. Now pick any $f \in D(B_\alpha)$ and consider the function

$$f_0 := f + af_+ + bf_-$$

and try to adjust the constants a and b so that $f_0(0) = f_0(1) = 0$, i.e., $f_0 \in D$. This amounts to solve the equations

$$af_+(0) + bf_-(0) + f(0) = 0 \text{ and } af_+(1) + bf_-(1) + f(1) = 0 \ .$$

We find

$$\begin{bmatrix} a \\ b \end{bmatrix} = -\frac{1}{f_+(0)f_-(1) - f_-(0)f_+(1)} \begin{bmatrix} f_-(1) & -f_-(0) \\ -f_+(1) & f_+(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$
$$= -f(0)\frac{1}{2}\frac{\sqrt{2}}{\sqrt{e^2-1}} \begin{bmatrix} e-\alpha \\ e\alpha - 1 \end{bmatrix}$$

which leads to

$$af_+ + bf_- = -f(0)\frac{e-\alpha}{e^2-1}[e^{-(x-1)} - \frac{\alpha e - 1}{\alpha - e} e^x] = c\frac{\sqrt{2}}{\sqrt{e^2-1}}[e^{-(x-1)} - \beta e^x] \ .$$

This shows that $D(A_\beta) = D(B_\alpha)$.

It is now very easy to compute the eigenvalues of B_β. It amounts to solving the equation

$$\frac{1}{i} f' = \lambda f$$

with the condition $f(1) = \beta f(0)$. The general solution is $ce^{i\lambda x}$ and the boundary condition requires

$$ce^{i\lambda} = c\beta$$

or

$$e^{i\lambda} = \beta$$

($c \neq 0$!) Write

$$\beta = e^{i\phi}$$

where $\phi \in [0, 2\pi)$ and then find that the eigenvalues are given by

$$\lambda_k = \phi + 2\pi k \ , \ \ k = 0, \pm 1, \pm 2, \dots \ .$$

The corresponding eigenfunctions are

$$e^{i(\phi + 2\pi k)x} \ .$$