PRACTICE FINAL EXAM

1. Curves

Problem 1: Find the parametric equations of the line that is tangent to the curve

$$\vec{r}(t) = (e^t, \sin t, \ln(1-t))$$

at $t = 0$.

$$\vec{r}(0) = (1, 0, 0) \ ,$$

$$\vec{r}'(t) = (e^t, \cos t, -\frac{1}{(1-t)}) \ , \vec{r}'(0) = (1, 1, -1)$$

and the equation of the line is

$$x(s) = 1 + s \ , y(s) = s \ , \ z(s) = -s \ .$$

Problem 2: Find the speed and the normal and tangential components of the acceleration and curvature for the curve $x(t) = \cos t, y(t) = \sin(t), z(t) = -t^2$.

Velocity:

$$\vec{v}(t) = \langle -\sin t, \cos t, -2t \rangle$$

Speed:

$$s'(t) = \sqrt{1 + 4t^2}$$

Acceleration:

$$\vec{a}(t) = \langle -\cos t, -\sin t, -2 \rangle$$

Tangential component:

$$a_T = s''(t) = \frac{4t}{\sqrt{1 + 4t^2}}$$

Normal component:

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2} = \frac{\sqrt{5 + 4t^2}}{\sqrt{1 + 4t^2}}$$

Curvature:

$$a_N = s'(t)^2 \kappa(t)$$

and hence

$$\kappa = \sqrt{5 + 4t^2} \sqrt{1 + 4t^2}$$

2. Optimization problems

Problem 3: Find the minimum cost area of a rectangular solid with volume 64 cubic inches, given that the top and sides cost 4 cents per square inch and the bottom costs 7 cents per square inch. Just set up the equations using Lagrange multipliers, you do not have to solve them.

The box has unknown dimensions x, y, z. The volume constraint is $xyz = 64$ and the area is $2(xy + xz + yz)$. The cost is $8(xy + xz) + 11yz$. We have minimize this cost given the constraint.

Lagrange multiplier leads to

$$8(y + z) = \lambda yz \ , 8x + 11z = \lambda xz \ , 8x + 11y = \lambda xy \ , xyz = 64 \ .$$

and we have to solve these equations for positive x, y, z.

(Actually this can be solved and yields $y = z = \frac{8}{(11)^{1/3}}, x = (11)^{2/3}$.)

Problem 4: Find the plane of the form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where $a, b, c > 0$ and that passes through the point $(2, 1, 4)$ and cuts off the smallest volume in the first octant.

The volume of the cutout region is

$$\frac{abc}{6}$$

which has to optimized over a, b, c given the constraint

$$\frac{2}{a} + \frac{1}{b} + \frac{4}{c} = 1 \ .$$

Lagrange:

$$\frac{bc}{6} = \lambda \frac{2}{a^2} \ , \frac{ac}{6} = \lambda \frac{1}{b^2} \ , \frac{ab}{6} = \lambda \frac{4}{c^2}$$

We find right away that $\lambda \neq 0$ and

$$\frac{2}{a} = \frac{1}{b} = \frac{4}{c} = \frac{1}{3}$$

because of the constraint. Hence $a = 6$, $b = 3$ and $c = 12$.

3. Double and triple integrals

Problem 5: Find the y moment of the first petal (mostly in the first quadrant) of the 3-leaf rose $r = \cos(3\theta)$. Just set up the integral (with limits) in polar coordinates. You do not have to evaluate it.

The cosine is positive for $-\pi/6 < \theta < \pi/6$, $(3\pi)/6 \leq \theta \leq (5\pi)/6$ and $(7\pi)/6 \leq \theta \leq (9\pi)/6$ Hence we take the limits $-\pi/6 < \theta < \pi/6$ and find the integral in polar coordinates using that $y = r\sin\theta$

$$\int_{-\pi/6}^{\pi/6} \int_0^{\cos(3\theta)} r\sin\theta r dr d\theta$$

(The integral actually vanishes. Just draw a picture.)

Problem 6: Compute the volume of the region that is bounded above by the plane $z = y$ and below by the paraboloid $z = x^2 + y^2$.

First we have to figure out the intersection of the plane with the paraboloid: $y = x^2 + y^2$ which leads to the equation $x^2 + y^2 - y = 0$. This is a circle given by $x^2 + (y - 1/2)^2 = 1/4$. Hence the base of the solid over which we integrate is a disk centered at $(0, 1/2)$ with radius $1/2$. The integral can be set up in terms of cylindrical coordinates: $x = r\cos\theta, y = r\sin\theta, z$. The disk is given by given by $0 \leq r \leq \sin\theta$ and $0 \leq \theta \leq \pi$. Hence we get

$$\int_0^{\pi} \int_0^{\sin\theta} \int_{r^2}^{r\sin\theta} dz r dr d\theta = \int_0^{\pi} \int_0^{\sin\theta} [r\sin\theta - r^2] r dr d\theta = \frac{1}{12} \int_0^{\pi} \sin^4\theta d\theta = \frac{\pi}{32}$$

4. Surface Integrals

Problem 7: Find the surface area of the parabolic cylinder $z = y^2$ that lies over the triangle with vertices $(0, 0), (0, 1), (1, 1)$ in the zy plane.

We can use the parametrization

$$\vec{r} = x\vec{i} + y\vec{j} + y^2\vec{k}$$

which leads to $\vec{r}_x = \vec{i}$, $\vec{r}_y = \vec{j} + 2y\vec{k}$ and $\vec{r}_x \times \vec{r}_y = \vec{k} - 2y\vec{j}$. Hence

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + 4y^2}\ .$$

The triangle with the given vertices can be written as the region in which $0 \leq x \leq y \leq 1$. Thus we have to compute

$$\int_0^1 \int_0^y \sqrt{1 + 4y^2} dx dy = \int_0^1 \sqrt{1 + 4y^2} y dy = \frac{1}{12}(1 + 4s)^{3/2}\Big|_0^1$$

Problem 8: Consider the surface $x^2 + y^2 + (z - 2)^2 = 4, 0 \leq z \leq 2$. (MISPRINT IN THE PROBLEM). Convert via Stokes' theorem the surface integral

$$\int_S \int \text{curl} F \cdot \vec{n} d\sigma$$

to a line integral. Here $\vec{F} = x^2y\vec{i} - xy^2\vec{j} + \sin z\vec{k}$. Set this line integral up, parametrize the curve, and reduce to an ordinary Calculus One integral with limits. Don't evaluate this integral.

The surface is the top half of the sphere. If we take the outward normal to the sphere then the curve bounding this region is the circle $x(t) = 2\cos t$, $y(t) = 2\sin t$, $z = 2$, $0 \le t \le 2\pi$. This yields the right orientation. The vector field evaluated on this curve is

$$\vec{F} = 8\cos^2 t \sin t\vec{i} - 8\sin^2 t \cos t\vec{j} + \sin 2\vec{k}$$

$$\vec{r}'(t) = -2\sin t\vec{i} + 2\cos t\vec{j}$$

so that

$$\int_0^{2\pi} \vec{F} \cdot \vec{r}'(t)dt = -\int_0^{2\pi} 32\cos^2 t \sin^2 t dt \ .$$

5. Line integrals and Stokes' Theorem

Problem 9: Compute the line integral of the vector field

$$\vec{F} = (xyz + 1, x^2 z, x^2 y)e^{xyz}$$

along the curve given in parametrized form by

$$\vec{r}(t) = (\cos t, \sin t, t) \ , \ 0 \le t \le \pi \ .$$

The curl of $\vec{F}$ vanishes. Hence it suffices to compute the line integral along any curve that connects the point $(1,0,0)$ with $(-1,0,\pi)$. We take the straight line $x(t) = 1 - 2t, y(t) = 0$ and $z(t) = t\pi, 0 \le t \le 1$. The tangent vector is

$$\langle -2, 0, \pi \rangle$$

and the field along this line is

$$\vec{F} = \langle 1, (1-2t)^2 t\pi, 0 \rangle$$

so that

$$\int_C \vec{F} \cdot d\vec{r} = -2$$

Problem 10: Compute the line integral $\int_C \vec{F} \cdot \vec{dr}$ where C is the curve given by the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = -y$, counterclockwise when viewed from above, and

$$\vec{F} = (x^2 + y, x + y, 4y^2 - z) \ .$$

The curl of $\vec{F}$ is $8y\vec{i}$. The next question is how to choose the surface with boundary C. We are going to choose it as the disk cut by the plane $z = -y$ from the sphere. The normal vector is

$$\vec{n} = \frac{1}{\sqrt{2}}\langle 0, 1, 1 \rangle = \frac{1}{\sqrt{2}}(\vec{j} + \vec{k})$$

and hence the dot product of $\vec{n}$ with $\vec{F}$ is zero. Thus $\int_C \vec{F} \cdot d\vec{r} = 0$.

6. Divergence Theorem

Problem 11: Use the divergence theorem to compute the outward flux of the vector field

$$\vec{F} = (x^2, y^2, z^2)$$

through the cylindrical can that is bounded on the side by the cylinder $x^2 + y^2 = 4$, bounded above by $z = 1$ and below by $z = 0$.

Again, we invoke an integral theorem, but this time the divergence theorem. One computes easily

$$\text{div}\vec{F} = 2(x + y + z)$$

and we have to integrate this over the cylinder. Using cylindrical coordinates

$$2\int_0^{2\pi}\int_0^2\int_0^1 [r(\cos\theta + \sin\theta) + z]dzrdrd\theta = 4\pi \ .$$

One can try to compute the flux directly. For the flux through the top one has to integrate

$$(x^2, y^2, 1) \cdot (0, 0, 1)$$

over the disk of radius 2, which yields 4π. The bottom disk is particularly easy since the normal vector is $(0, 0, -1)$ and the vector field is $(x^2, y^2, 0)$ so that the dot product vanishes. Hence there is no contribution. It remains to compute the flux through the side. The parametrization of the cylinder is

$$\vec{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$$

so that

$$\vec{r}_\theta = (-2\sin\theta, 2\cos\theta, 0) \ , \ \vec{r}_z = (0, 0, 1)$$

and

$$\vec{r}_\theta \times \vec{r}_z = 2(\cos\theta, \sin\theta, 0)$$

which obviously points outward. Now

$$\vec{F} \cdot \vec{n}d\sigma = ((2\cos\theta)^2, (2\sin\theta)^2, z^2) \cdot 2(\cos\theta, \sin\theta, 0)d\theta dz = 8((\cos\theta)^3 + (\sin\theta)^3)d\theta dz$$

and

$$8\int_0^1\int_0^{2\pi} ((\cos\theta)^3 + (\sin\theta)^3)d\theta dz = 0 \ .$$

Problem 12: Compute the flux of $\vec{F} = 5zy^3\vec{i} + xz\vec{j} + 3z\vec{k}$ through the surface $x^2 + y^2 + z^2 = 9$ using the divergence theorem.

One computes $\text{div}\vec{F} = 3$ and the volume of the ball with radius 3 equals $\frac{4\pi}{3}3^3 = 36\pi$ and hence we get for the flux

$$108\pi \ .$$