NAME:

SECTION:

TEST 2 FOR MATH 2551 F1-F4, NOVEMBER 7, 2018

IMPORTANT: WRITE YOUR NAME AND SECTION NUMBER ON EVERY PAGE!

This test should be taken without any notes and calculators. Time: 50 minutes. Show your work and write legibly otherwise credit cannot be given. If you realize that you have written something which is wrong then, please, cross it out.

An integral that might be useful:

$$\int_{-\pi/2}^{\pi/2} (\cos\theta)^4 d\theta = \frac{3\pi}{8}$$

Problem 1:

Problem 2:

Problem 3:

Problem 4:

Problem 5:

Problem 6:

Total:

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Problem 1: a) (5 points) Find the gradient of the function $f(x, y) = \frac{1}{1+x^2+y^2}$.

Solution:

$$\nabla f(x, y) = -\frac{2}{(1 + x^2 + y^2)^2} \langle x, y \rangle \ .$$

b) (10 points) Find the unit vector $\vec{u}$ that points in the direction of fastest increase at the point $(1, 2)$ for the function f given above.

Solution:

$$\nabla f(1, 2) = -\frac{1}{18} \langle 1, 2 \rangle$$

and hence

$$\vec{u} = -\frac{\langle 1, 2 \rangle}{\sqrt{5}}$$

Problem 2: a) (10 points) Compute the equation of the plane tangent to the graph of the function $f(x, y) = x^4 + x^2y^2$ at the point $(2, 1, 20)$.

Solution:

$$\nabla f(x, y) = \langle 4x^3 + 2xy^2, 2x^2y \rangle$$
$$\nabla f(2, 1) = \langle 36, 8 \rangle$$

$$z = 20 + 36(x - 2) + 8(y - 1)$$

b) (5 points) Write the line normal to the plane tangent obtained in a) in parametric form.

Solution: Direction $\langle 36, 8, -1 \rangle$, point on line $2, 1, 20)$.

$$x = 2 + 36t \ , \ y = 1 + 8t \ , \ z = 20 - t \ .$$

WRITE ONLY WORK RELATED TO PROBLEMS 1 AND 2 ON THIS PAGE

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Problem 3: a) (5 points) Consider the function $f(x, y) = x^2 - xy + y^2$. Find all the points of local extrema.

Solution: We have $f_x = 2x - y = 0$ and $f_y = 2y - x = 0$ and it follows that $(0, 0)$ is the only critical point.

b) (15 points) Find the absolute minima and absolute maxima of the function f given in part a) on the closed triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 4$ and $y = x$.

Solution: The only critical point is $(0, 0)$ which is on the boundary. Hence we have to check the boundary for maxima and minima.

a) $y = x, 0 \le x \le 4$: The function to investigate is $g(x) = x^2$ which has a max at $x = 4$ and min at $x = 0$. Which yields the potential candidates $(0, 0)$ and $(4, 4)$.

b) $y = 4, 0 \le x \le 4$: The function to investigate is $g(x) = x^2 - 4x + 16$. $g'(x) = 2x - 4 = 0$ and hence we have new possible candidates $(0, 4), (2, 4)$.

c) $x = 0, 0 \le y \le 4$: The function to investigate is $g(y) = y^2$ which has a max at $y = 4$ and a min at $y = 0$ which does not give any new point.

$f(0, 0) = 0$, $f(4, 4) = 16$, $f(0, 4) = 16$, $f(2, 4) = 10$

Problem 4: (20 points) Find the points on the curve $x^2 - 8x + y^2 + 15 = 0$ that are closest and farthest to the origin.

Solution: Have to minimize $x^2 + y^2$ subject to the constraint $x^2 - 8x + y^2 + 15 = 0 = 0$. Using Lagrange multipliers:

$$2x = \lambda(2x - 8) \ , \ 2y = \lambda 2y \ ,$$

or

$$x(1 - \lambda) = -4\lambda \ , \ y(1 - \lambda) = 0 \ .$$

From the first equation $\lambda \neq 1$ and hence $y = 0$. The $x = -\frac{4\lambda}{1-\lambda}$ and the constraint yields

$$0 = \frac{16\lambda^2}{(1 - \lambda)^2} + \frac{32\lambda}{1 - \lambda} + 15$$

or

$$16\lambda^2 + 32\lambda(1 - \lambda) + 15(1 - \lambda)^2 = 0$$

which simplifies to

$$(1 - \lambda)^2 = 16$$

which has the solutions $\lambda_1 = 5$ and $\lambda_2 = -3$. Hence we have the points $(5, 0)$ the farthest and $(3, 0)$ the closest.

WRITE ONLY WORK RELATED TO PROBLEMS 3 AND 4 ON THIS PAGE

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Problem 5: (15 points) Integrate the function $f(x, y) = xy$ over the region between the curves $y = x^2$ and $y = 2x$.

Solution: Intersection point is $(2, 2)$. Slice vertically.

$$\int_0^2 \int_{x^2}^{2x} xydydx = \int_0^2 x\frac{y^2}{2}\Big|_{x^2}^{2x} dx = \int_0^2 x(2x^2 - \frac{1}{2}x^4)dx = \frac{1}{2}x^4 - \frac{1}{12}x^6\Big|_0^2 = \frac{8}{3}$$

Problem 6: (20 points) Use polar coordinates to integrate the function $f(x, y) = x^2 + y^2$ over the region bounded by the circle $(x - 1)^2 + y^2 = 1$.

Solution: In polar coordinates the equation of the circle is $r = 2\cos\theta, -\pi/2 \leq \theta \leq \pi/2$. Hence the integral is

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 rdrd\theta = \int_{-\pi/2}^{\pi/2} \frac{r^4}{4}\Big|_0^{2\cos\theta} d\theta = 4\int_{-\pi/2}^{\pi/2} (\cos\theta)^4 d\theta = \frac{3\pi}{2}$$

WRITE ONLY WORK RELATED TO PROBLEMS 5 AND 6 ON THIS PAGE