HOMEWORK 1, DUE JANUARY 26 IN CLASS

Problem 1: (This problem uses facts from real analysis) Let $u \in W^{1,p}(0,1)$ for $1 \leq p \leq \infty$. Show that u equals almost everywhere an absolutely continuous function v and its weak derivative u' equals the pointwise derivative v' almost everywhere. (Hint: Pick $b > a \in (0,1)$ and arbitrary. Consider the function $\psi_\varepsilon(x) = \phi_\varepsilon(x-b) - \phi_\varepsilon(x-a)$, set $\eta_\varepsilon = \int_0^x \psi_\varepsilon(z)dz$, use the definition of the weak derivative and let ε go to zero. Here ϕ_ε is a non-negative 'bump' function, i.e., smooth with support in $(-\varepsilon, \varepsilon)$ and $\int \phi_\varepsilon(x)dx = 1$)

Problem 2: a) Prove the inequality

$$\|f\|_\infty^2 \leq \|f\|_{L^2(\mathbb{R})} \|f'\|_{L^2(\mathbb{R})}$$

for all functions in $C_c^1(\mathbb{R})$. (Hint: Write $f(x)^2 = 2\int_{-\infty}^x ff'dx$ and also $f(x)^2 = -2\int_x^\infty ff'dx$ and use Schwarz's inequality.)

b) Is there a function, not necessarily in $C_c^1(\mathbb{R})$, that yields equality?

Problem 3: Fix any point $x_0 \in \mathbb{R}$ and consider the linear functional $\ell(\phi) = \phi(x_0)$ where $\phi \in C_c^\infty(\mathbb{R}$.

a) Show that ℓ can be uniquely extended to a bounded linear functional on $H^1(\mathbb{R})$.

b) Show that there exists a unique $u_0 \in H^1(\mathbb{R})$ such that $(u_0, v)_{H^1(\mathbb{R})} = \ell(v)$ for all $v \in H^1(\mathbb{R})$ and check that $u_0(x) = e^{-|x-x_0|}$.

Problem 4: A function $u : \mathbb{R}^n \to \mathbb{R}$ is Hölder continuous of order $0 < \alpha < 1$ if

$$\|u\|_{C^\alpha} := \|u\|_\infty + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty \ .$$

This space $C^\alpha(\mathbb{R})$ is a Banach space. Show that any function $u \in W^{1,p}(\mathbb{R})$ for some $1 \leq p < \infty$ is almost everywhere equal to a function that is Hölder continuous of order $\alpha = 1 - \frac{1}{p}$. (Hint: Prove the estimate

$$\|u\|_{C^\alpha} \leq C\|u\|_{W^{1,p}(\mathbb{R})}$$

for functions $u \in C_c^1(\mathbb{R})$ and then use the fact that these functions are dense in $W^{1,p}(\mathbb{R})$.)