HOMEWORK 1, DUE JANUARY 26 IN CLASS

Problem 1: (This problem uses facts from real analysis) Let $u \in W^{1,p}(0,1)$ for $1 \le p \le \infty$. Show that u equals almost everywhere an absolutely continuous function v and its weak derivative u' equals the pointwise derivative v' almost everywhere. (Hint: Pick $b > a \in (0,1)$ and arbitrary. Consider the function $\psi_\varepsilon(x) = \phi_\varepsilon(x-b) - \phi_\varepsilon(x-a)$, set $\eta_\varepsilon = \int_0^x \psi_\varepsilon(z)dz$, use the definition of the weak derivative and let ε go to zero. Here ϕ_ε is a non-negative 'bump' function, i.e., smooth with support in $(-\varepsilon, \varepsilon)$ and $\int \phi_\varepsilon(x)dx = 1$)

Solution: Using the hint, we consider

$$u \star \phi_\varepsilon(b) - u \star \phi_\varepsilon(a) = \int_0^1 u\psi_\varepsilon = -\int_0^1 u'\eta_\varepsilon \ .$$

Since $u \in L^p(0,1)$ there exists a sequence $\varepsilon_j \to 0$ such that the left side converges to $u(b)-u(a)$ for almost every a, b whereas η_ε converges pointwise a.e. to the characteristic function of the interval $[a,b]$. Hence, by dominated convergence we have that for a.e. $a, b \in (0,1)$

$$u(b) - u(a) = \int_a^b u'dx \ .$$

Pick a such that the above holds for almost every b. The function $v(b) = \int_a^b u'dx$ is absolutely continuous (why?) and hence $u = v$ almost everywhere. By the integration by parts formula, which holds for absolutely continuous functions, we have for any $\phi \in C_c^\infty(0,1)$

$$\int v'\phi dx = -\int v\phi' dx = -\int u\phi' dx = \int u'\phi dx$$

and hence $u' = v'$ a.e.

Problem 2: a) Prove the inequality

$$\|f\|_\infty^2 \le \|f\|_{L^2(\mathbb{R})}\|f'\|_{L^2(\mathbb{R})}$$

for all functions in $C_c^1(\mathbb{R})$. (Hint: Write $f(x)^2 = 2\int_{-\infty}^x ff'dx$ and also $f(x)^2 = -2\int_x^\infty ff'dx$ and use Schwarz's inequality.)

b) Is there a function, not necessarily in $C_c^1(\mathbb{R})$, that yields equality?

Solution: Using the hint we find that

$$2f(x)^2 \le 2\int_{-\infty}^\infty |f||f'|dx$$

which by Schwarz's inequality leads to the desired conclusion. To solve b) consider the function $g(x) = e^{-a|x|}$ where $a > 0$ is a constant. Clearly

$$\|g\|_\infty^2 = 1$$

and

$$\|g\|_2^2 = \frac{1}{a}, \ \ \|g'\|_2^2 = a$$

and hence there is equality. One can show that any function that yields equality is of the form

$$Ce^{-a|x-b|}$$

where $a > 0, b, C > 0$ are constants.

Problem 3: Fix any point $x_0 \in \mathbb{R}$ and consider the linear functional $\ell(\phi) = \phi(x_0)$ where $\phi \in C_c^\infty(\mathbb{R}$.

a) Show that ℓ can be uniquely extended to a bounded linear functional on $H^1(\mathbb{R})$.

b) Show that there exists a unique $u_0 \in H^1(\mathbb{R})$ such that $(u_0, v)_{H^1(\mathbb{R})} = \ell(v)$ for all $v \in H^1(\mathbb{R})$ and check that $u_0(x) = e^{-|x-x_0|}$.

Solution: Using the previous result we find that for any $\phi \in C_c^\infty(\mathbb{R})$ we have that

$$|\ell(\phi)| = |\phi(x_0)| \le [\|\phi\|_{L^2(\mathbb{R})}\|f\phi'\|_{L^2(\mathbb{R})}]^{1/2} \le \frac{1}{\sqrt{2}}[\|\phi\|^2_{L^2(\mathbb{R})} + \|f\phi'\|^2_{L^2(\mathbb{R})}]^{1/2} = \frac{1}{\sqrt{2}}\|\phi\|_{H^1(\mathbb{R})}$$

hence ℓ can be extended to a bounded linear functional on all of H^1 since $C_c^\infty(\mathbb{R})$ is dense. Likewise, the extension is unique because $C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$. For, if ℓ_1 and ℓ_2 are two extensions then for any $u \in H^1(\mathbb{R})$ we may pick a sequence $\phi_k \in C_c^\infty(\mathbb{R})$ converging to u in $H^1(\mathbb{R})$ and hence

$$|\ell_1(u) - \ell_2(u)| \le |\ell_1(u) - \ell_1(\phi_k)| + |\ell_2(u) - \ell_2(\phi_k)| + |\ell_1(\phi_k) - \ell_2(\phi_k)|$$

and since the last term vanishes and the others tend to zero we have that $\ell_1(u) = \ell_2(u)$. This is an instance of the fact that any bounded linear operator defined on a dense set can be uniquely extended as a bounded operator. By the Riesz representation theorem there exists a unique $u_0 \in H^1(\mathbb{R})$ such that

$$\ell(v) = \int (u_0 v + u_0' v')dx \ .$$

Now we integrate by parts and we see that

$$\int [-v'' + v]u_0 dx = v(x_0)$$

where $v \in \mathbb{C}_c^\infty(\mathbb{R})$. consider v with supp $v \subset (x_0, \infty)$ then if we choose $u_0 = Ae^{-(x-x_0)}$ we see by integration by parts that

$$\int [-v'' + v]u_0 dx = \int [-u_0'' + u_0]v dx = 0 \ .$$

The same holds if v has support on the left of x_0 but we have to choose $u_0 = Ae^{(x-x_0)}$ in other words our candidate is

$$u_0(x) = Ae^{-|x-x_0|} \ .$$

Now we compute

$$\int [u_0 v + u_0' v']dx = \int_{-\infty}^{x_0} [u_0 - u_0'']v dx + u_0' v\Big|_{-\infty}^{x_0} + \int_{x_0}^{\infty} [u_0 - u_0'']v dx + u_0' v\Big|_{x_0}^{\infty} = 2Av(x_0)$$

and hence $\frac{1}{2}e^{-|x-x_0|}$ is the unique function that does the job.

Problem 4: A function $u : \mathbb{R}^n \to \mathbb{R}$ is Hölder continuous of order $0 < \alpha < 1$ if

$$\|u\|_{C^\alpha} := \|u\|_\infty + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty \ .$$

This space $C^\alpha(\mathbb{R})$ is a Banach space. Show that any function $u \in W^{1,p}(\mathbb{R})$ for some $1 \leq p < \infty$ is almost everywhere equal to a function that is Hölder continuous of order $\alpha = 1 - \frac{1}{p}$. (Hint: Prove the estimate

$$\|u\|_{C^\alpha} \leq C\|u\|_{W^{1,p}(\mathbb{R})}$$

for functions $u \in C_c^1(\mathbb{R})$ and then use the fact that these functions are dense in $W^{1,p}(\mathbb{R})$.)

Solution: For $u \in C_c^1(\mathbb{R})$ we have by Hölder's inequality

$$u(x)^p = p\int_{-\infty}^x u^{p-1}u' dx \leq p\|u'\|_p \|u^{p-1}\|_q \leq p\|u'\|_p\|u\|_p^{p-1}$$

since $q = \frac{p}{p-1}$. In a further step we use the fundamental theorem of calculus and estimate

$$|u(a) - u(b)| \leq \int_b^a |u'|dx \leq |b-a|^{\frac{1}{q}}\|u'\|_p$$

where once more $q = \frac{p}{p-1}$. Hence, with $\alpha = \frac{p-1}{p}$ we get that

$$\frac{|u(a) - u(b)|}{|b-a|^\alpha} \leq \|u'\|_p \leq \|u\|_{W^{1,p}(\mathbb{R})} \ .$$

We are basically done. We may improve the presentation by using Young's inequality which says that for any positive numbers a, b we have that

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

so that

$$\|u'\|_p\|u\|_p^{p-1} \leq \frac{1}{p}\|u'\|_p^p + \frac{p-1}{p}\|u\|_p^p$$

and hence

$$u(x)^p \leq \|u'\|_p^p + (p-1)\|u\|_p^p \leq p\|u\|_{W^{1,p}(\mathbb{R})}^p$$

which shows that

$$\|u\|_{C^\alpha} \leq (1 + p^{1/p})\|u\|_{W^{1,p}(\mathbb{R})} \ . \tag{1}$$

If $u \in W^{1,p}(\mathbb{R})$, there exists a sequence u_k of functions in $C_c^1(\mathbb{R})$ such that

$$\|u - u_k\|_{W^{1,p}(\mathbb{R})} \to 0$$

as $k \to \infty$ and hence u_k is a Cauchy sequence in $C^\alpha(\mathbb{R})$ by (**??**) and hence converges to some function v. Note that this convergence is uniform and hence for any $\phi \in C_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} u\phi dx = \lim_{k\to\infty}\int_{\mathbb{R}} u_k\phi dx = \int_{\mathbb{R}} v\phi dx$$

and we conclude that $u = v$ almost everywhere.