HOMEWORK 2, DUE FEBRUARY 9 IN CLASS

Problem 1: a) Let U be open. Prove the inequality

$$\|Du\|^2_{L^2(U)} \leq \|u\|_{L^2(U)} \|\Delta u\|_{L^2(U)}$$

for $u \in C_c^\infty(U)$.

b) Let U be open, bounded and ∂U be C^∞. Prove that the inequality continuous to hold for all $u \in H^2(U) \cap H_0^1(U)$. (Hint: A sequence in $C_c^\infty(U)$ can be used to prove the inequality for $u \in H_0^2(U)$. That is easy! Use an approximating sequence $u_k \in C^\infty(\overline{U})$ to prove it for $u \in H^2(U) \cap H_0^1(U)$. Note, that the trace theorem could be useful in this context as well as Gauss's theorem.)

Problem 2: Suppose that U is open and connected and that $u \in W^{1,p}(U)$ with

$$Du = 0$$

a.e. in U. Show that u is constant a.e. in U. (Hint: mollify!)

Problem 3: Let $F : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that F' is bounded. Suppose further that $U \subset \mathbb{R}^n$ is a bounded domain and that $u \in W^{1,p}(U)$ for some $1 \leq p \leq \infty$. Show that $F(u) \in W^{1,p}(U)$ and that as weak derivatives

$$F(u)_{x_i} = F'(u) u_{x_i} \ , \ i = 1, \ldots, n \ .$$

(Hint: Use approximations by smooth function and that any sequence that converges in L^p has a subsequence that converges pointwise a.e.)

Problem 4: Assume that U is bounded. Use Problem 3 and the function

$$F_\varepsilon(z) = \begin{cases} (z^2 + \varepsilon^2)^{1/2} - \varepsilon & \text{if} z \geq 0 \\ 0 & \text{if} z < 0 \end{cases}$$

to show that for any function $u \in W^{1,p}(U)$, $u_+(x) = \max(u(x), 0)$ is in $W^{1,p}(U)$ and that

$$Du_+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \leq 0\} \end{cases} .$$

(Hint: Do not forget the dominated convergence theorem.)