HOMEWORK 2, DUE FEBRUARY 9 IN CLASS

Problem 1: a) Let U be open. Prove the inequality

$$\|Du\|^2_{L^2(U)} \leq \|u\|_{L^2(U)}\|\Delta u\|_{L^2(U)}$$

for $u \in C_c^\infty(U)$.

Solution: We have that

$$\int_U |Du|^2 dx = -\int_U u\Delta u dx \leq \|u\|_{L^2(U)}\|\Delta u\|_{L^2(U)}$$

using Schwarz's inequality.

b) Let U be open, bounded and ∂U be C^∞. Prove that the inequality continuous to hold for all $u \in H^2(U) \cap H^1_0(U)$. (Hint: First use a sequence in $C_c^\infty(U)$ to prove the inequality for $u \in H_0^2(U)$. Then use an approximating sequence $u_k \in C^\infty(\overline{U})$ to prove it for $u \in H^2(U) \cap H^1_0(U)$. Note, that the trace theorem is useful in this context.)

Solution: Let $u \in H_0^2(U)$. There exists $u_k \in C_c^\infty(U)$ convergent to u in $H^2(U)$. Because

$$\int_U |Du|^2 dx = \lim_{k\to\infty} \int_U |Du_k|^2 dx \leq \lim_{k\to\infty} \|u_k\|_{L^2(U)}\|\Delta u_k\|_{L^2(U)} = \|u\|_{L^2(U)}\|\Delta u\|_{L^2(U)}$$

the inequality follows for all $u \in H_0^2(U)$. Now, let $u \in H^2(U) \cap H^1_0(U)$. There exists $u_k \in C^\infty(\overline{U})$ such that u_k converges to u in $H^2(U)$. The function u is also in $H^1_0(U)$ and hence the trace Tu_k converges to Tu in $L^2(\partial U)$ which, however, equals to zero a.e. on ∂U. Hence $Tu_k \to 0$ in $L^2(\partial U)$. Since ∂U is smooth, we may use Gauss's theorem

$$\int_U |Du_k|^2 dx = -\int_U u_k \Delta u_k dx + \int_{\partial U} u_k N \cdot Du_k dS$$

where N is the outward normal. Now

$$\left|\int_{\partial U} u_k N \cdot Du_k dS\right| \leq \|u_k\|_{L^2(\partial U)}\|Du_k\|_{L^2(\partial U)} .$$

Since $u \in H^2(U)$ we know that $u_{x_i} \in H^1(U)$ and hence it has a trace in $L^2(U)$. Thus,

$$\|u_k\|_{L^2(\partial U)}\|Du_k\|_{L^2(\partial U)} \leq \|u_k\|_{L^2(\partial U)}\|u_k\|_{H^2(U)}$$

which converges to 0 as $k \to \infty$. By Schwarz's inequality as before

$$\left|-\int_U u_k \Delta u_k dx\right| \leq \|u_k\|_{L^2(U)}\|\Delta u_k\|_{L^2(U)}$$

and as $k \to \infty$ the desired inequality emerges.

Problem 2: Suppose that U is open and connected and that $u \in W^{1,p}(U)$ with

$$Du = 0$$

a.e. in U. Show that u is constant a.e. in U.

Solution: Consider the domain $U_\varepsilon = \{x \in U : \text{dist}(x, \partial U) > \varepsilon\}$. For $x \in U_\varepsilon$ consider the function $u_\varepsilon(x) = \int \eta_\varepsilon(x-y)u(y)dy$ where η_ε is a standard molifier. $u_\varepsilon \in C^\infty(U_\varepsilon)$ and

$$Du_\varepsilon(x) = \int \eta_\varepsilon(x-y)Du(y)dy$$

by the definition of the weak derivative. Sine $Du = 0$ a.e., we have that $Du_\varepsilon(x) = 0$ for all $x \in U_\varepsilon$ and hence u_ε is constant in U_ε. Since $\|u_\varepsilon - u\|_{L^p(U_\varepsilon)} \to 0$ as $\varepsilon \to 0$, we have that u is constant a.e..

Problem 3: Let $F : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that F' is bounded. Suppose further that $U \subset \mathbb{R}^n$ is a bounded domain and that $u \in W^{1,p}(U)$ for some $1 \le p \le \infty$. Show that $F(u) \in W^{1,p}(U)$ and that as weak derivatives

$$F(u)_{x_i} = F'(u)u_{x_i} \ , \ i = 1, \dots, n \ .$$

(Hint: Use that any sequence that converges in L^p has a subsequence that converges pointwise a.e.)

Solution: Let $u \in W^{1,p}(U)$. There exists $u_k \in C^\infty(U) \cap W^{1,p}(U)$ such that

$$\|u - u_k\|_{W^{1,p}(U)} \to 0$$

as $k \to \infty$. Now

$$|F(u_k)(x) - F(u_\ell)(x)| \le C|u_k(x) - u_\ell(x)|$$

since F' is bounded. Moreover

$$DF(u_k)(x) = F'(u_k)(x)Du_k(x) \ .$$

From this it follows that for any test function $\phi \in C_c^\infty(U)$

$$\int F(u)D\phi dx = \lim_{k\to\infty} \int F(u_k)D\phi dx = -\lim_{k\to\infty} \int F'(u_k)Du_k\phi dx \ .$$

We may choose a subsequence so that u_k tends a.e. to u and hence $F'(u_k)$ a.e to $F'(u)$. Now

$$|\int [F'(u_k)Du_k - F'(u)Du]\phi dx| = |\int [F'(u_k) - F'(u)]Du_k + F'(u)[Du_k - Du]\phi dx|$$

$$\le C \int |F'(u_k) - F'(u)|^2 |\phi dx + \|F'(u)\|_\infty \|u_k - u\|_{W^{1,p}(U)}$$

which tends to zero. Hence

$$\int F(u)D\phi dx = -\int F'(u)Du\phi dx \ .$$

Since U is bounded $F(u) \in L^p(U)$ and since $F'(u)$ is bounded $F'(u)u_{x_i} \in L^p(U)$ and hence $F(u) \in W^{1,p}(U)$.

Problem 4: Assume that U is bounded. Use Problem 3 and the function

$$F_\varepsilon(z) = \begin{cases} (z^2+\varepsilon^2)^{1/2} - \varepsilon & \text{if} z \geq 0 \\ 0 & \text{if} z < 0 \end{cases}$$

to show that for any function $u \in W^{1,p}(U)$, $u_+(x) = \max(u(x), 0)$ is in $W^{1,p}(U)$ and that

$$Du_+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \leq 0\} \end{cases} .$$

Solution: The function F_ε is continuously differentiable, in fact its derivative is

$$F'_\varepsilon(z) = \begin{cases} \frac{z}{\sqrt{z^2+\varepsilon^2}} & \text{for } z > 0 \\ 0 & \text{for } z \leq 0 \end{cases}$$

which is bounded by 1 and continuous. Moreover, we have that

$$F_\varepsilon(z) \leq |z|$$

and

$$\lim_{\varepsilon \to 0} F_\varepsilon = \max(z, 0) .$$

Hence by the previous problem, for every $\varepsilon > 0$ $F_\varepsilon(u) \in W^{1,p}(U)$ and its weak derivative is given by

$$DF_\varepsilon(u) = F'_\varepsilon(u) Du .$$

Hence for any test function ϕ using dominated convergence

$$\int u_+ D\phi dx = \lim_{\varepsilon \to 0} \int F_\varepsilon(u) D\phi dx = -\lim_{\varepsilon \to 0} \int F'_\varepsilon(u) Du\phi dx = -\lim_{\varepsilon \to 0} \int_{u>0} F'_\varepsilon(u) Du\phi dx$$

which, again by dominated convergence equals

$$-\int_{u>0} Du\phi dx .$$

Hence

$$\int Du_+ \phi dx = \int_{u>0} Du\phi dx$$

or

$$Du_+ = \chi_{u>0} Du$$

and the statement follows.