HOMEWORK 3, DUE MARCH 14 IN CLASS

Problem 1: Let $U \subset \mathbb{R}^n$ be open bounded with smooth boundary and

$$Lu = -\sum_{i,j}^{n} (a^{ij} u_{x_i})_{x_j} + cu \ .$$

Assume that a^{ij} and c are smooth and a^{ij} satisfiy the uniform ellipticity condition. Prove that

$$B[u,v] := \sum_{ij} \int_U a^{ij} u_{x_i} v_{x_j} dx + \int_U cuvdx + \mu \int_U uvdx$$

$u, v \in H_0^1(U)$ satisfies the hypotheses of the Lax-Milgram theorem provided that $c(x) \geq -\mu$ in U.

Problem 2: Assume that $U \subset \mathbb{R}^n$ is open bounded with smooth boundary and connected. A function $u \in H^1(U)$ is a weak solution of the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial U \end{cases}$$

if

$$\int_U Du \cdot Dv dx = \int_U fv dx$$

all $v \in H^1(U)$. For $f \in L^2(U)$ prove that this problem has a weak solution if and only if $\inf_U f dx = 0$.

Problem 3: Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of the semilinear equation

$$-\Delta u + c(u) = f$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth, $c(0) = 0$ and $c'(x) \geq 0$. Prove that $u \in H^2(\mathbb{R}^n)$.

Problem 4: With the usual assumptions on U further assume that U is connected. Use energy methods to show that the only smooth solutions of the Neumann problem

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial U \end{cases}$$

are the constant functions. Repeat this problem using maximum principles.