HOMEWORK 4, DUE APRIL 6 IN CLASS

Problem 1: Prove that the sequence $u_k(x) = \sin(kx)$ converges weakly to zero as $k \to \infty$ in $L^2(0,1)$.

Problem 2: Let U be an open, bounded subset of $\mathbb{R}^n$ with smooth boundary. Prove the existence of a nontrivial weak solution $u \in H_0^1(U)$ of

$$\begin{cases} -\Delta u = |u|^{q-1}u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

for $1 < q < \frac{n+2}{n-2}$. (Hint: Interpret this equation as the Euler-Lagrange equation of a functional.)

Problem 3: Let U be an open, bounded subset of $\mathbb{R}^n$ with smooth boundary and denote by U_T the set $U \times (0,T]$. In U_T consider the PDE

$$u_t - \Delta u - \varepsilon u_{tt} = 0 \ .$$

Here $\varepsilon > 0$. Show that this PDE is the Euler-Lagrange equation of an energy functional of the form

$$\int_{U_T} dxdt L_\varepsilon(Dw, w_t, w, x, t) \ .$$

(Hint: Look for a Lagrangian with an exponential term involving t.)

Problem 4: Let U be an open, bounded subset of $\mathbb{R}^2$ with smooth boundary and let $\Sigma \subset \mathbb{R}^3$ be the graph of the smooth function $u : U \to \mathbb{R}$. Show that

$$\int_U (1 + |Du|^2)^{-3/2} \det D^2 u dx$$

depends only on the values of Du on the boundary ∂U. This integral represents the Gauss curvature of the graph Σ.