HOMEWORK 5, DUE APRIL 18 IN CLASS

Problem 1: a) Assume that $n \geq 3$. Find a constant c such that

$$u(x) = (1 + |x|^2)^{\frac{2-n}{2}}$$

is a solution of the equation

$$-\Delta u = c u^{\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n$$

b) Check that for each $\lambda > 0$

$$u_\lambda(x) := \lambda^{\frac{n-2}{2}} u(\lambda x)$$

is also a solution.

c) Show that

$$\|u_\lambda\|_{L^{2^*}(\mathbb{R}^n)} = \|u\|_{L^{2^*}(\mathbb{R}^n)} \ , \|Du_\lambda\|_{L^2(\mathbb{R}^n)} = \|Du\|_{L^2(\mathbb{R}^n)}$$

for each $\lambda > 0$ and hence $\{u_\lambda\}$ is not precompact in $L^{2^*}(\mathbb{R}^n)$.

Problem 2: Let u be a solution of the semilinear heat equation

$$u_t - \Delta u = f(u) \text{ in } \mathbb{R}^n \times (0, \infty) \ .$$

Assume that u and its derivatives go to zero rapidly as $|x| \to \infty$.

a) Show that

$$\frac{d}{dt} \int_{\mathbb{R}^n} \frac{1}{2}|Du|^2 - F(u)dx = -\int_{\mathbb{R}^n} u_t^2 dx$$

where $F' = f, F(0) = 0$.

b) Show that

$$\frac{d}{dt} \int_{\mathbb{R}^n} |x|^2 \left(\frac{1}{2}|Du|^2 - F(u) \right) dx = -\int_{\mathbb{R}^n} u_t^2 |x|^2 - 2nF(u) + (n-2)|Du|^2 dx \ .$$

This is a parabolic analogue of the Derrick-Pohozaev identity.

Problem 3: Let $a : \mathbb{R} \to \mathbb{R}$ be continuous and assume that for any sequence $f_n \in L^2(0,1)$ that converges weakly to f we have that $a(f_n)$ converges weakly to $a(f)$ in $L^2(0,1)$. Show that a must be of the form $a(z) = \alpha z + \beta$ where α and β are constants.