EXISTENCE OF SOLUTIONS FOR THE HEAT EQUATION BASED ON THE HILLE-YOSIDA THEOREM

The problem is the prove existence of solutions to the initial value problem

$$\begin{cases} u_t + Lu = 0 & \text{in } U \times (0,T] \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\} \end{cases} \tag{1}$$

where $g \in L^2(U)$. We make the usual assumptions such as the uniform ellipticity condition and that U is bounded and ∂U is smooth.

$$Lu = -\sum_{i,j}(a^{i,j}u_{x_i})_{x_j} + \sum_i b^i u_{x_i} + cu \ ,$$

and we assume that the coefficients of L are also smooth.

Recall that

$$B[u,v] = \sum_{i,j}\int_U a^{i,j}u_{x_i}v_{x_j} + \sum_i b^i u_{x_i} v + cuv dx$$

and also recall that

$$B_\gamma[u,v] := B[u,v] + \gamma(u,v)_{L^2(U)}$$

for $\gamma > 0$ sufficiently large satisfies the coercivity condition

$$\beta\|u\|^2_{H^1_0(U)} \le B_\gamma[u,u]$$

for some positive constant β. We think of (1) formally as constructing the semi group e^{-Lt} but the problem is that this semi group is not contractive, because $B[u,u]$ is not positive in general. Hence we solve the problem

$$\begin{cases} u_t + (L+\gamma)u = 0 & \text{in } U \times (0,T] \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\} \end{cases} \tag{2}$$

One easily checks that if $u(x,t)$ is a solution of (2) then $e^{\gamma t}u(x,t)$ solves the original problem (1).

Here is the Hille Yosida Theorem:

Theorem 0.1. *A closed and densely defined linear operator A on some Banach space X is the generator of a contraction semi-group if and only if*

$$(0,\infty) \subset \rho(A) \quad \text{and} \quad \|R_\lambda\|_X \le \frac{1}{\lambda} \ , \ \lambda > 0 \ .$$

We apply the Hille-Yosida Theorem and prove

Theorem 0.2. *Let $D(A) = H^1_0(U) \cap H^2(U)$ and define for $u \in D(A)$*

$$Au = -(L+\gamma)u \ .$$

Then A is the generator of a contraction semi group on $L^2(U)$.

Proof. The Banach space X is in our case the Hilbert space $L^2(U)$. We have seen before before that $D(A)$ is dense in $L^2(U)$. We check that A is closed. Let $u_k \in D(A)$ be a sequence that converges to u in $L^2(U)$ and such that Au_k converges to v in $L^2(U)$. Trivially we can interpret u_k as a weak solution of the boundary value problem

$$\begin{cases} (L+\gamma)u_k = -Au_k & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

and from the chapter on regularity we know that there exists a constant C depending only on the coefficients of L and on the domain U such that

$$\|u_k - u_\ell\|_{H^2(U)} \le C(\|Au_k - Au_\ell\|_{L^2(U)} + \|u_k - u_\ell\|_{L^2(U)}) \ .$$

This implies that u_k is a Cauchy sequence in $H^2(U)$ and hence converges to some function $u \in H^2(U)$ in $H^2(U)$. Since $u_k \in H_0^1(U)$ it also follows that $u \in H_0^1(U)$ and hence $u \in D(A)$. The convergence in $H^2(U)$ also implies that Au_k converges to Au in $L^2(U)$ and thus, A is closed. Pick any $\lambda > 0$ and $f \in L^2(U)$ and consider the problem of finding a weak solution of

$$\begin{cases} (L+\gamma+\lambda)u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} .$$

We know that the assumptions of the Lax-Milgram lemma are satisfied and hence there exists a unique weak solution $u \in H_0^1(U)$ for this problem. More precisely

$$B_\gamma[u,v] + \lambda(u,v)_{L^2(U)} = (f,v)_{L^2(U)}$$

for all $v \in H_0^1(U)$. Thus, we can formally at least denote the solution by

$$u = (\lambda I - A)^{-1} f \ .$$

From elliptic regularity theory we know that $u \in H^2(U)$ and hence $u \in D(A)$. Thus, $(\lambda I - A)(D(A)) = L^2(U)$ and $\lambda I - A$ is one-to-one and onto. We also have that

$$\beta \|u\|^2_{H_0^1(U)} \le B_\gamma[u,u]$$

and thus,

$$(\beta+\lambda)\|u\|^2_{L^2(U)} \le B_\gamma[u,u] + \lambda\|u\|^2_{L^2(U)} = (f,u)_{L^2(U)} \ .$$

so that

$$(\beta+\lambda)\|u\|^2_{L^2(U)} \le \|f\|_{L^2(U)}\|u\|_{L^2(U)}$$

and hence

$$\|u\|_{L^2(U)} \le \frac{1}{(\beta+\lambda)}\|f\|_{L^2(U)} < \frac{1}{\lambda}\|f\|_{L^2(U)} \ .$$

Hence $(0,\infty) \subset \rho(A)$ and $\|R_\lambda\| \le \frac{1}{\lambda}$ and the conditions of the Hille-Yosida Theorem are verified.

□