The Sobolev inequality, a general uncertainty principle

The uses of the Coulomb Uncertainty principle are restricted to problems related to the hydrogenic atom. For more general potentials $V(x)$ the Sobolev inequality serves as a very effective uncertainty principle.

Theorem 1: Sobolev's inequality *For $n \geq 3$ let f be a function in $C^1(R^n)$ with compact support. Then there exists a constant C_n depending only on the dimension but not on f so that*

$$\|f\|_p \leq S_n \|\nabla f\|_2$$

where

$$p = \frac{2n}{n-2} \ ,$$

which is called the **Sobolev index**

REMARK 1: Note that inequality requires $n \geq 3$. It does not make a statement in 2 and 3 dimensions.

REMARK 2: The value of the Sobolev index can be understood as follows. Assuming that the inequality holds, pick any function f and consider its scaled verion $f(\lambda x)$ with $\lambda > 0$ arbitrary. Then, by changing variables

$$\left(\int_{R^n} |f(\lambda x)|^p dx \right)^{1/p} = \lambda^{-n/p} \left(\int_{R^n} |f(x)|^p dx \right)^{1/p}$$

which is

$$\leq C_n \left(\int_{R^n} |\nabla(f(\lambda x))|^2 dx \right)^{1/2} = \lambda^{1-n/2} C_n \left(\int_{R^n} |\nabla(f(x))|^2 dx \right)^{1/2} \ .$$

Thus, the λ powers must necessarily be the same, i.e., $n/p = n/2 - 1$.

REMARK 3: The best possible constant in Sobolev's inequality is known and it has the value.

$$\frac{n(n-2)}{4} |S^n|^{2/n}$$

where $|S^n|$ is the surface area of the unit n-sphere in R^{n+1}, i.e.,

$$|S^n| = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})} \ .$$

The functions which yield equality are of the form

$$\frac{\text{const.}}{(\mu^2 + |x-a|^2)^{(n-2)/2}} \ .$$

This result is due to Talenti [T] and Aubin[A] and its proof is somewhat more involved. See also [L] and [CL] for other proofs.

Exercise: For which values of p is it possible for the inequality

$$\|f\|_p \le C_{n,q} \|\nabla f\|_q \ , \tag{1}$$

to hold.

ANSWER:

$$p = \frac{qn}{n-q} \ .$$

In particular for $q = 1$, $p = n/(n-1)$.

PROOF: We present the standard proof found in the textbooks, which is due to Gagliardo and Nirenberg, and prove the more general inequality (1). In order to present the ideas as clearly as possible we do it in 3-space and leave the general argument as an exercise.

Using the fundamental theorem of calculus

$$f(x,y,z) = \int_{-\infty}^{x} \partial_x f(r,y,z) dr$$

and in particular

$$|f(x,y,z)| \le \int_{-\infty}^{\infty} |\partial_x f(r,y,z)| dr =: g_1(y,z) \ .$$

Similarly, repeating the same argument in the other variables

$$|f(x,y,z)|^3 \le g_1(y,z) g_2(x,z) g_3(x,y) \ ,$$

and hence

$$\|f\|_{3/2} \le \left(\int \sqrt{g_1(y,z)} \sqrt{g_2(x,z)} \sqrt{g_3(x,y)} dx dy dz \right)^{2/3} \ .$$

Using Schwarz' inequality on the x- variable yields the upper bound

$$\left(\int \sqrt{g_1(y,z)} \sqrt{\int g_2(x,z) dx} \sqrt{\int g_3(x,y) dx} dy dz \right)^{2/3}$$

Applying Schwarz' inequality once more in the y-variable yields

$$\left(\int \sqrt{\int g_1(y,z) dy} \sqrt{\int g_2(x,z) dx} \sqrt{\int g_3(x,y) dx dy} dz \right)^{2/3} \ ,$$

and finally in the z-variable

$$\left(\sqrt{\int g_1(y,z) dy dz} \sqrt{\int g_2(x,z) dx dz} \sqrt{\int g_3(x,y) dx dy} \right)^{2/3} \ ,$$

$$= \left(\int g_1(y,z) dydz \int g_2(x,z) dxdz \int g_3(x,y) dxdy \right)^{1/3} ,$$
$$= \left(\|\partial_x f\|_1 \|\partial_y f\|_1 \|\partial_z f\|_1 \right)^{1/3} ,$$
$$\leq \|\nabla f\|_1 .$$

Thus we have established that

$$\|f\|_{3/2} \leq \|\nabla f\|_1 . \tag{2}$$

To arrive at the general inequality, replace f by $|f|^s$ for a number $s > 0$ to be chosen later and calculate

$$\|f^s\|_{3/2} \leq s \||\nabla f||f|^{s-1}\|_1$$

Using Hölder's inequality on the right side yields the estimate

$$\|f^s\|_{3/2} \leq s \||\nabla f|\|_q \||f|^{s-1}\|_{q'} \tag{3}$$

where $1/q + 1/q' = 1$ or $q' = q/(q-1)$. Now if we choose $s = 2q/(3-q)$ so that

$$3s/2 = (s-1)q/(q-1) = \frac{3q}{3-q} = p ,$$

we get from (3)

$$\|f\|_p^{2p/3} \leq 2q/(3-q) \||\nabla f|\|_q \|f\|_p^{p(q-1)/q}$$

and upon dividing both sides by $\|f\|_p^{p(q-1)/q}$ we obtain

$$\|f\|_p^{2p/3 - p(q-1)/q} \leq 2q/(3-q) \||\nabla f|\|_q ,$$

which is our desired inequality. Note, as a check, that

$$p[2/3 - (q-1)/q] = 1 .$$

Exercise: By setting up a careful induction argument prove inequality (2) in any dimension. Then preceed to prove (1) for all $1 \leq q < n$,

REMARK: The sharp constant in (2) is strongly related to the isoperimetric inequality. This is a substantial subject all by itself and we just touch it with a few remarks. The inequality (2) on R^n in its sharp form reads as

$$\|f\|_{\frac{n}{n-1}} \leq n^{\frac{-(n-1)}{n}} |S^{n-1}|^{-1/n} \|\nabla f\|_1 .$$

In other words, we claim that

$$sup_{f \neq 0} \frac{\|f\|_{\frac{n}{n-1}}}{\|\nabla f\|_1} = n^{\frac{-(n-1)}{n}} |S^{n-1}|^{-1/n} .$$

The constant is precisely the surface area of a ball divided by the $(n-1)/n$-th power of its volume. The constant is not attained by any function whose gradient is integrable but we can get arbitrarily close as the following calculation shows. Define the function $f_\varepsilon(x) = u_\varepsilon(|x|)$ where

$$u_\varepsilon(r) = \begin{cases} 1 & \text{for } r < 1 \\ 0 & \text{for } r > 1+\varepsilon \\ \frac{1+\varepsilon-r}{\varepsilon} & \text{for } 1 \le r \le 1+\varepsilon \end{cases} .$$

We have immediately (please check) that $\lim_{\varepsilon\to 0} \|f_\varepsilon\|_{n/(n-1)} = (|S^{n-1}|/n)^{(n-1)/n}$. Next $\nabla f_\varepsilon = u'_\varepsilon(x/|x|)$ where

$$u'_\varepsilon(r) = \begin{cases} 0 & \text{for } r < 1 \\ 0 & \text{for } r > 1+\varepsilon \\ \frac{1}{\varepsilon} & \text{for } 1 \le r \le 1+\varepsilon \end{cases} .$$

Hence, using polar coordinates (please check)

$$\|\nabla f_\varepsilon\|_1 = |S^{n-1}| \int_1^{1+\varepsilon} \frac{1}{\varepsilon} r^{n-1} dr = |S^{n-1}| \frac{1}{n} \frac{1}{\varepsilon} \left[(1+\varepsilon)^n - 1\right]^n \ ,$$

which tends to $|S^{n-1}|$ in the limit as $\varepsilon \to 0$. Hence we get from this example that the sharp constant

$$C_n \ge n^{-(n-1)/n} |S^{n-1}|^{-1/n} \ .$$

That $C_n \le n^{-(n-1)/n} |S^{n-1}|^{-1/n}$ is much more difficult to see. One way of getting at it is using the co-area formula. Imagine that f is a nice positive, smooth function, that has no flat spots, i.e., ∇f vanishes only at isolated points, the critical points. Thus, the level surfaces $\{x : f(x) = \alpha\}$ consist either of critical points or otherwise are $n-1$ dimensional surfaces perpendicular to ∇f which does not vanish on these surfaces.

For any given funcition g we shall rewrite the integral

$$\int g(x) |(\nabla f)(x)| dx$$

'using f as a variable'. Imagine a point on $\{x : f(x) = \alpha\}$ the level surface of f at height α. Pick a small cube of volume '$(\Delta x)^n$' by choosing $n-1$ edges of length $\Delta s_1, \dots, \Delta s_{n-1}$ tangential and one edge of length Δp perpendicular to the surface. The change in f along the perpendicular edge is up to an error of higher order $|\Delta f| = |\nabla f| \Delta p$ and hence

$$(\Delta x)^n = \frac{1}{|\nabla f|} |\Delta f| \Delta s_1 \cdots \Delta s_{n-1} \ .$$

Note that $\Delta s_1 \cdots \Delta s_{n-1}$ corresponds to the surface area element and hence we can write

$$dx = \frac{1}{|\nabla f|} d\alpha dS$$

and

$$\int g(x) |(\nabla f)(x)| dx = \int_0^\infty d\alpha \int_{\{x: f(x) = \alpha\}} g(x) dS \ , \tag{4}$$

where $d\alpha$ is the change in height of the level surface and dS is the area element on the level surface $\{x : f(x) = \alpha\}$. For a rigorous proof of this formula see [BZ]. Equation (4) is known as the co-area formula. If one replaces the measure dS by the Hausdorff measure then the co-area formula holds in great generality for Sobolev functions, i.e., functions whose weak derivative is p-summable for some p. In particular we have that

$$\int |(\nabla f)(x)| dx = \int_0^\infty d\alpha \int_{\{x: f(x)=\alpha\}} dS = \int_0^\infty d\alpha |\{x : f(x) = \alpha\}| \ , \tag{5}$$

where $|\{x : f(x) = \alpha\}|$ is the surface area of the level set.

Thus we have now some geometric understanding of the L^1 norm of the gradient. Let us emphasize that these considerations are somewhat heuristic but can be made rigorous. They belong properly to geometric measure theory.

Let us try to write the $\int |f(x)|^p dx$ in a similar fashion. Start with

$$|f(x)| = \int_0^{|f(x)|} d\alpha = \int_0^\infty \chi_{\{|f(x)|>\alpha\}}(x) d\alpha \ , \tag{6}$$

where $\chi_A(x)$ is the characteristic function of the set A, i.e., it is equals 1 if $x \in A$ and equals 0 if $x \notin A$. It is a straightforward computation to see that

$$\int |f(x)|^p dx = p \int_0^\infty \alpha^{p-1} |\{x : |f(x)| > \alpha\}| d\alpha \ , \tag{7}$$

where $|A|$ denotes Lebesgue measure of the set A. In essence this is a possible definition of the Lebesgue integral of the function $|f(x)|^p$.

As a consequence we see that the L^p norm of a function is entirely determined by the volume of the regions that are enclosed by the level surfaces $\{x : |f(x)| = \alpha\}$.

From (5) and (7) we can draw an interesting conclusion. The equation (5) says that the L^1-norm of ∇f depends only on the surface area of the level surfaces. Hence it is natural to try to minimize these areas but keeping the volumes fixed. Using the isoperimetric inequality the best arrangement is to deform the level sets $\{x : |f(x)| > \alpha\}$ into balls centered at some common point, say the origin and choosing the radius in such a way that the volume of these balls is the same as $|\{x : |f(x)| > \alpha\}|$. To these rearranged level sets corresponds also a function, which is called f^* the *symmetric decreasing rearrangement* of f. This function has the value α on the boundary of the open ball whose volume is $|\{x : |f(x)| > \alpha\}|$.

Returning to our Sobolev inequality (2), but in R^n, we see among all functions the spherically symmetric functions deliver the worst constant. Thus, we may assume that all the level sets are rearranged into balls with radius

$$\left[\frac{n}{|S^{n-1}|}\right]^{\frac{1}{n}} |\{x : |f(x)| > \alpha\}|^{1/n}$$

and hence this inequality reads

$$C_n \geq \left[\frac{1}{n-1}\right]^{\frac{n-1}{n}} |S^{n-1}|^{-1/n} \sup_{f\neq 0} \frac{\left[\int_0^\infty \alpha^{1/(n-1)}\lambda(\alpha)^{\frac{n}{n-1}} d\alpha\right]^{\frac{n-1}{n}}}{\int_0^\infty \lambda(\alpha) d\alpha} \tag{8}$$

where $\lambda(\alpha) = |\{x : |f(x)| > \alpha\}|^{\frac{n-1}{n}}$. Two observations about the function $\lambda(\alpha)$: it is a non increasing function and we may assume that $\int_0^\infty \lambda(\alpha)d\alpha = 1$ as well as $\lambda(0) = 1$, since the scaling $\lambda(\alpha) \to C\lambda(D\alpha)$ leaves the ratio in (8) fixed. To maximize

$$\left[\int_0^\infty \alpha^{1/(n-1)}\lambda(\alpha)^{\frac{n}{n-1}} d\alpha\right]^{\frac{n-1}{n}}$$

over all such functions $\lambda(\alpha)$ we proceed as follows. The functional

$$\lambda(\alpha) \mapsto \mathcal{F}(\lambda) = \left[\int_0^\infty \alpha^{1/(n-1)}\lambda(\alpha)^{\frac{n}{n-1}} d\alpha\right]^{\frac{n-1}{n}}$$

is convex. Now restrict the set over which to maximize to consist of non-increasing functions that have the value 1 at $\alpha = 0$, whose integral equals 1 and are zero outside the interval $[0, N]$ for some N large. Call this set T_N and note that T_N is a convex set and

$$F(N) = \sup_{\lambda\in T_N} \mathcal{F}(\lambda)$$

is non decreasing as a function of N.

Since our functional is convex it attains its maximum on the set T_N at the extreme points which consists of functions that take only zero and 1 as values. Since the function is non-increasing, has the value 1 at $\alpha = 0$ and integrates to 1 it must be

$$\lambda_{opt}(\alpha) = \chi_{[0,1]}(\alpha)\ . \tag{9}$$

which does not depend on the value of N as long as $N > 1$. and hence inserting this into (8) we have that

$$C_n \geq |S^{n-1}|^{-1/n} n^{-(n-1)/n}\ ,$$

which demonstrates our claim.

References:

[A] Aubin, T., Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non-lineaire pour la transformation conforme de la courbure scalaire, J. Funct. Anal. **57** (1979), 148-174.

[BZ] Brothers, J. and Ziemer, W. P., Minimal rearrangements of Sobolev functions, J. Reine Angew. Math. **384** (1988), 153-179.

[CL] Carlen, E.A. and Loss, M. Extremals of functionals with competing symmetries, J. Funct. Anal. **88** (1990), 194-211.

[L] Lieb, E.h., sharp constants in the nHardy-Littlewood-Sobolev and related inequalities, Ann. Math. **118** (1983), 349-374.

[T] Talenti, G., Best constant in Sobolev inequality, Ann. Mat. Pura Appl., **110** (1976),353-372.