Some remarks about Sobolev spaces

We have been talking about functions that are square integrable and have a square integrable gradient. In this context the standard notion of differentiability of a function is not adequate. The key property of any space of functions in analysis is completeness and while it is possible to construct Banach spaces using the usual definition of the derivative, they are not easy to handle. \( L^p \) spaces are fine, since it is easy to identify bounded linear functionals via the Riesz representation theorem.

A notion of derivative that is much better adapted to our purposes is the notion of weak derivative.

**Definition** A function \( f \in L^2(\mathbb{R}^n) \) is in \( H^1(\mathbb{R}^n) \) if there exist \( n \) functions \( g_i \in L^2(\mathbb{R}^n) \) such that for all \( \phi \in C_c^\infty(\mathbb{R}^n) \)

\[
\int_{\mathbb{R}^n} f(x) \frac{\partial \phi}{\partial x_i}(x)dx = -\int_{\mathbb{R}^n} g_i(x)\phi(x)dx
\]

for all \( i = 1, 2, \ldots n. \)

Of course we should think of the functions \( g_i \) as the partials of \( f \) and we shall use this notation. However, please remember that \( \frac{\partial f}{\partial x_i} \) is defined as the function that satisfies the infinite set of equations given by

\[
\int_{\mathbb{R}^n} f(x) \frac{\partial \phi}{\partial x_i}(x)dx = -\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}(x)\phi(x)dx
\]

Note that the function \( \frac{\partial f}{\partial x_i} \) (x) is uniquely defined since the set \( C_c^\infty(\mathbb{R}^n) \) is dense in \( L^2(\mathbb{R}^n) \).

We can endow the set \( H^1(\mathbb{R}^n) \) with the inner product

\[
(u, v)_{H^1} := \int_{\mathbb{R}^n} u(x)v(x)dx + \int_{\mathbb{R}^n} \nabla u(x)\nabla v(x)dx
\]

The wonderful thing about the notion of weak derivative is that \( H^1(\mathbb{R}^n) \) is a Hilbert space, i.e., it is complete. To see this pick a Cauchy sequence \( u_j \). Since \( u_j \) is a Cauchy sequence in \( L^2 \) it converges to an element \( u \in L^2(\mathbb{R}^n) \). Likewise, the partials \( \partial u_j/\partial x_i \) converge to functions \( v_i \in L^2(\mathbb{R}^n) \). All we have to show that \( v_i \) is the weak derivative of \( u \). Using Schwarz’ inequality we learn that

\[
\lim_{j \to \infty} \int_{\mathbb{R}^n} u_j(x) \frac{\partial \phi}{\partial x_i}(x)dx = \int_{\mathbb{R}^n} u(x) \frac{\partial \phi}{\partial x_i}(x)dx ,
\]

and

\[
\lim_{j \to \infty} \int_{\mathbb{R}^n} \frac{\partial u_j}{\partial x_i}(x)\phi(x)dx = \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i}(x)\phi(x)dx .
\]

Since

\[
\int_{\mathbb{R}^n} u_j(x) \frac{\partial \phi}{\partial x_i}(x)dx = -\int_{\mathbb{R}^n} \frac{\partial u_j}{\partial x_i}(x)\phi(x)dx
\]
the same holds for $u$.

A sequence of functions $u_j$ converges to $u$ weakly in $H^1(\mathbb{R}^n)$ if for every bounded linear functions $L$, $\lim_{j \to \infty} (u_j) = L(u)$. Using the Riesz representation theorem, an equivalent formulation is that $(u_j, v)_{H^1} \to (u, v)_{H^1}$. One of the really important theorems is

**Theorem:** Every bounded sequence in $H^1(\mathbb{R}^N)$ has a weakly convergent subsequence, i.e., for every $u_j \in H^1(\mathbb{R}^n)$, with $\|u_j\|_{H^1} \leq C$ there exists $u \in H^1(\mathbb{R}^n)$ and a subsequence $u_{j_k}$ so that $u_{j_k} \to u$ weakly in $H^1(\mathbb{R}^n)$.

All the results about the Sobolev space $H^1(\mathbb{R}^n)$ so far have been on an abstract level. Nowhere did we make any specific use of the derivative. This will change dramatically in the next theorem.

**Theorem:** Rellich-Kondrachev Let $u_j$ be a bounded sequence in $H^1(\mathbb{R}^n)$ which we can assume to converge weakly to the function $u \in H^1(\mathbb{R}^n)$. Then for any measurable set $B$ with finite measure and any $q < p = 2n/(n-2)$ we have that

$$
\int_B |u_j(x) - u(x)|^q dx \to 0
$$
as $j \to \infty$.

**PROOF:** Quite generally for $f \in H^1(\mathbb{R}^n)$ we have that for any $h \in \mathbb{R}^n$

$$
\int |f(x+h) - f(x)|^2 dx \leq \|\nabla f\|^2 |h|^2 .
$$

(1)

A simple way to see this is to use Plancherel’s theorem to calculate

$$
\int |f(x+h) - f(x)|^2 dx = \int |e^{2\pi i p \cdot h} - 1|^2 |\hat{f}(p)|^2 dp \leq 4\pi^2 \int |p|^2 |\hat{f}(p)|^2 dp |h|^2 = \|\nabla f\|^2 |h|^2 ,
$$

where the inequality $|e^{2\pi i p \cdot h} - 1| \leq 2\pi |p| |h|$ has been used. Further for any smooth function $\phi$ of compact support with $\int \phi(y) dy = 1$ and $\int |\phi(y)| |y| dy < \infty$ we calculate

$$
\|f * \phi - f\|_2 \leq \int |\phi(y)| |y| dy \|\nabla f\|_2 ,
$$

(2)

where

$$(f * \phi)(x) = \int f(x-y)\phi(y) dy .$$

To see this write

$$
\|f * \phi - f\|_2 = \left[ \int |\int f(x-y)\phi(y) dy - \int f(x)\phi(y) dy|^2 dx \right]^{1/2} .
$$
Minkowski’s inequality allows to pull the $y$ integration outside the $x$ integration to yield the bound
\[ \int |\phi(y)||f(\cdot - y) - f(\cdot)|_2 \]
which together with (1) proves (2).

Now we consider the sequence $u_j$ and pick $\varepsilon$ arbitrary but fixed. Set
\[ \phi_m(y) = m^n \phi(ym) \]
and note that \( \int \phi_m(y) |y|dy = 1 \) and
\[ \int |\phi_m(y)||y|dy = \frac{1}{m} \int |\phi(y)||y|dy. \]
Since \( \|\nabla u_j\| \leq C \) we have uniformly in \( j \) that
\[ \|u_j \ast \phi_m - u_j\|_2 \leq \frac{1}{m} C \int |\phi(y)||y|dy < \varepsilon, \]
for some fixed \( m \) sufficiently large. This follows from (2).

Next since $u_j \rightharpoonup u$ weakly in $H^1(R^n)$ and hence in $L^2(R^n)$ we get that
\[ u_j \ast \phi_m(x) \rightharpoonup u \ast \phi_m(x) \]
for every $x$. Moreover
\[ |u_j \ast \phi_m(x)| \leq \|u_j\|_2 \|\phi_m\|_2 \leq C' \]
uniformly in \( j \). Thus we conclude using the dominated convergence theorem that
\[ \int_B |u_j \ast \phi_m(x) - u \ast \phi_m(x)|^2dx \to 0 \]
as \( j \to \infty \). Hence
\[ \|u_j - u\|_{L^2(B)} \leq \|u_j - u_j \ast \phi_m\|_{L^2(B)} + \|u_j \ast \phi_m - u \ast \phi_m\|_{L^2(B)} + \|u_j \ast \phi_m - u\|_{L^2(B)} < 2\varepsilon + \|u \ast \phi_m - u\|_{L^2(B)} \]
and using (30 we conclude that
\[ \limsup_{j \to \infty} \|u_j - u\|_{L^2(B)} < 2\varepsilon. \]
Since $\varepsilon$ is arbitrary this proves the theorem for $q = 2$ and since $B$ is bounded it holds for all $1 \leq q \leq 2$.

For $q < p$ we use Hölder’s inequality
\[ \|u_j - u\|_q \leq \|u_j - u\|_2^\theta \|u_j - u\|_p^{1-\theta} \]
where $\theta = n(1/q - 1/p) > 0$. By Sobolev’s inequality
\[ \|u_j - u\|_p \leq S_n(\|\nabla u_j\|_2 + \|\nabla u\|_2) \leq 2CS_n \]
and the general theorem follows from the case $q = 2$. 

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