Improved constants

In recent years there have been further results concerning optimal constants in the Lieb-Thirring inequality. As it was mentioned before, the most important inequalities concerning stability of matter are the estimates on the sum of the eigenvalues and also on the sum of square roots of the eigenvalues. Without going too much into details we mention here an approach due to Laptev and Weidl [LW] using matrix valued potentials. We consider Schrödinger operators of the form

\[ H = -\frac{d^2}{dx^2} \otimes I - U(x) \]

where \( U(x) \) is for every \( x \in \mathbb{R} \) a positive hermitean \( n \times n \) matrix and \( I \) is he \( n \times n \) identity matrix. Recall that an hermitean matrix \( A \) is positive if for every vector \( x \in \mathbb{C}^n \)

\[ \langle x, Ax \rangle \geq 0 \]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( \mathbb{C}^n \). The operator \( H \) acts on vectors where each component is a function of the variable \( x \). We assume that the potential matrix is nice in the sense that the entries are smooth functions with compact support. This is not really a restriction but avoids technicalities. We consider the operator \( H \) on \( L^2(\mathbb{R}; \mathbb{C}^n) \). Like in the case of a scalar valued potential we consider the eigenvalues and arrange them in increasing order. The lowest eigenvalue is defined by minimizing

\[
\int \langle \psi'(x), \psi'(x) \rangle dx - \int \langle \psi(x), U(x)\psi(x) \rangle dx
\]

subject to the constraint

\[
\int \langle \psi(x), \psi(x) \rangle dx = 1
\]

it is easy to see that a minimizer \( \psi_0(x) \) exists. The next eigenvalue is found by optimizing orthogonal to \( \psi_0 \), etc. In this way we get eigenvalues \( -\lambda_1 \leq -\lambda_2 \leq \cdots \). The following theorem was proved in [LW] (see also [BL]).

**Theorem:** For the operator \( H \) we have the inequality

\[
\sum_j \lambda_j^{3/2} \leq \frac{3}{16} \int \text{Tr}[U(x)^2] dx
\]

Alternatively, the above estimate can be written as

\[
\sum_j \lambda_j^{3/2} \leq \frac{1}{2\pi} \int \text{Tr}[p^2 \otimes I - U(x)^2] - dpdx
\]

which displays the semiclassical nature of the bound. Following Aizenman and Lieb [AL] we can deduce from this bound a whole series of sharp bounds as follows. Write for \( \gamma > 3/2 \)

\[
\lambda^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(5/2)\Gamma(\gamma - 3/2)} \int_0^\infty (\lambda - s)^{3/2} s^{\gamma - 3/2} ds = \frac{\Gamma(\gamma + 1)}{\Gamma(5/2)\Gamma(\gamma - 3/2)} \int_0^\infty (\lambda - s)^{3/2} s^{\gamma - 3/2} ds
\]
From this we get immediately

**Theorem** For the operator $H$ we have for all $\gamma \geq 3/2$

$$\sum_{j} \lambda_{j}^{\gamma} \leq \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 3/2)} \int \text{Tr}[U(x)^{\gamma+1/2}] dx$$

or alternatively

$$\sum_{j} \lambda_{j}^{\gamma} \leq \frac{1}{2} \int \text{Tr}[p^2 \otimes I - U(x)^{2}] \gamma dp dx$$

As an application we give a sketch of the proof of the theorem of Laptev and Weidl.

**Theorem: Laptev and Weidl** On $L^2(R^n)$ consider the negative eigenvalues $-\lambda_1 < -\lambda_2 \leq \cdots$ of the operator $-\Delta - U(x)$ where $U$ is non-negative. Then for all $\gamma \geq 3/2$

$$\sum_{j} \lambda_{j}^{\gamma} \leq \frac{1}{(2\pi)^n} \int [p^2 - U(x)]^{\gamma} dx dp .$$

This is a semiclassical estimate and best possible.

**PROOF:** Following Laptev and Weidl we use induction on the dimension. We have to estimate

$$\text{Tr}[\gamma - \Delta - U(x)] \gamma \Gamma \text{Tr}[\gamma - \Delta' - U(x_1, x')] \gamma$$

where $x' = x_2, \ldots, x_n$. The right side is estimated from above by

$$\text{Tr}[\gamma - \Delta' - U(x_1, x')] \gamma$$

it is not hard to see that

$$[\gamma - \Delta' - U(x_1, x')]_-$$

is for every $x_1$ a positive, compact operator and hence can be approximated by finite matrices. Applying our theorem, noting that $\gamma \geq 3/2$ we obtain

$$\text{Tr}[\gamma - \Delta - U(x)] \gamma \Gamma \text{Tr}[\gamma - \Delta' - U(x_1, x')] \gamma + 1/2 \leq \frac{1}{2\pi} \int \gamma [p^2 \otimes I - \gamma - \Delta' - U(x_1, x')] \gamma + 1/2 dx_1 ,$$

where the trace is preformed over the remaining variables $x_2, \ldots x_n$. Since

$$\frac{1}{2\pi} \int \gamma [p^2 \otimes I' - \gamma - \Delta' - U(x_1, x')] \gamma + 1/2 dx_1 = \text{const.} \int \text{Tr}[\gamma + 1/2] dx_1 \gamma + 1/2$$

we can apply the theorem once more since $\gamma + 1/2 > 3/2$ and get that

$$\int \text{Tr}[\gamma - \Delta' - U(x_1, x')] \gamma + 1/2 dx_1 \leq \frac{1}{2\pi} \int \gamma [p^2 \otimes I'' - \gamma - \Delta'' - U(x_1, x_2, x'')] \gamma + 1/2 dx_2 dx_1 .$$
so that
\[
\frac{1}{2\pi} \text{Tr} \int [p_1^2 \otimes I' - [-\Delta' - U(x_1, x')]_+] \gamma^{1/2} dp_1 dx_1
\]
\[
\leq \frac{1}{(2\pi)^2} \text{Tr} \int [(p_1^2 + p_2^2) \otimes I'' - [-\Delta'' - U(x_1, x', x'')] - dp_1 dp_2 dx_1 dx_2 .
\]
Repeating this procedure leads eventually to the desired estimate.

Another sharp bound is the following, due to Hundertmark, Lieb and Thomas in the scalar case and Hundertmark, Laptev and Weidl in the matrix case

**Theorem** For the operator $H$ we have the inequality
\[
\sum_j \lambda_j^{1/2} \leq \frac{1}{2} \int \text{Tr}[U(x)] dx .
\]

**PROOF:** We apply the Birman-Schwinger principle. $-\lambda$ is an eigenvalue if and only if 1 is an eigenvalue of
\[
U(x)^{1/2} \frac{1}{-\partial^2 + \lambda} U(y)^{1/2} .
\]
The associated kernel is given by
\[
U(x)^{1/2} e^{-\sqrt{\lambda}|x-y|/2\sqrt{\lambda}} U(y)^{1/2} = \frac{1}{\sqrt{\lambda}} \mathcal{L}_\lambda .
\]
Hence whenever the Birman Schwinger kernel has an eigenvalue 1 the operator $\mathcal{L}_\lambda$ has an eigenvalue $\sqrt{\lambda}$. denote by $e_j(\lambda)$ the eigenvalues of the operator $\mathcal{L}_\lambda$ ordered decreasingly. By what we said above $e_1(\lambda_1) = \sqrt{\lambda_1}, e_2(\lambda_2) = \sqrt{\lambda_2} \ldots$ We shall prove that
\[
\sum_{i=1}^k e_i(\lambda)
\]
is a decreasing function of $\lambda$ for all $k$. Once this is established we have that
\[
\sum_j \sqrt{\lambda_j} = \sum_j e_j(\lambda_j) \leq e_1(\lambda_2) + e_2(\lambda_2) + e_3(\lambda_3) + \ldots
\]
\[
\leq e_1(\lambda_3) + e_2(\lambda_3) + e_3(\lambda_3) + \sum_{j=4} e_j(\lambda_j)
\]
etc. so that
\[
\sum_j \sqrt{\lambda_j} \leq \text{Tr}\mathcal{L}_{\lambda=0} = \frac{1}{2} \int \text{Tr}U(x) dx .
\]
Note that the first trace is a trace over $L^2 \otimes C^n$ while the second is a trace over $C^n$ only.
To establish the monotonicity we write

\[ \mathcal{L}_\lambda = U(x)^{1/2} \sqrt{\frac{\lambda}{\pi}} \int \frac{1}{p^2 + \lambda} e^{ip(x-y)} dp \, dx \]

The kernel

\[ G_\varepsilon(p) = \frac{\varepsilon}{\pi} \frac{1}{p^2 + \varepsilon^2} \]

is the Poisson kernel and it is easy to check that

\[ G_\varepsilon \ast G_\varepsilon' = G_{\varepsilon + \varepsilon'} \]

and by a simple calculation

\[ \mathcal{L}_\lambda(x, y) = \int dq e^{iqx} \mathcal{L}_{\lambda-\varepsilon^2}(x, y) e^{-iqy} \frac{\varepsilon}{\pi} \frac{1}{q^2 + \varepsilon^2} \]

In other words the operator can be written as an average of the following form

\[ \mathcal{L}_\lambda = \int w(q) U_q \mathcal{L}_{\lambda-\varepsilon^2} U_q^* \]

where \( U_q \) is unitary and \( \int w(q) dq = 1 \). Let \( P_k \) be the projector onto the space belonging to the \( k \) largest eigenvalues. Thus,

\[ \sum_{j=1}^{k} e_j(\lambda) = \text{Tr} P_k \mathcal{L}_\lambda P_k = \int w(q) \text{Tr} [P_k U_q \mathcal{L}_{\lambda-\varepsilon^2} U_q^* P_k]\]

\[ = \int w(q) \text{Tr} [U_q^* P_k U_q \mathcal{L}_{\lambda-\varepsilon^2} U_q^* U_q P_k U_q] \]

by the cyclicity of the trace. The operator \( U_q^* P_k U_q \) is again a projection of dimension \( k \) and hence by the minimax principle

\[ \text{Tr} [U_q^* P_k U_q \mathcal{L}_{\lambda-\varepsilon^2} U_q^* P_k U_q] \leq \sum_{j=1}^{k} e_j(\lambda - \varepsilon^2) \]

Therefore

\[ \sum_{j=1}^{k} e_j(\lambda) \leq \sum_{j=1}^{k} e_j(\lambda - \varepsilon^2) \]

which is what we had to prove.

We apply now these theorems to prove LT inequalities with good constants. They are not optimal but close.
Theorem: On $L^2(R^3)$ consider the negative eigenvalues $-\lambda_1 < -\lambda_2 \leq \cdots$ of the operator $\Delta - U(x)$ where $U$ is non-negative. Then

$$\sum_j \lambda_j^{1/2} \leq \frac{1}{8\pi} \int U(x)^2 \, dx$$

and

$$\sum_j \lambda_j \leq \frac{2}{15\pi^2} \int U(x)^{5/2} \, dx.$$ 

which is twice the semiclassical constant.

PROOF: We proceed as in the previous proof concerning the powers $\gamma \geq 3/2$. We write for $\gamma \geq 1/2$

$$\lambda^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(3/2)\Gamma(\gamma - 1/2)} \int_0^\infty (\lambda - \alpha)^{1/2} \alpha^{\gamma - 1/2} \frac{d\alpha}{\alpha}$$ \hspace{1cm} (1)

Applying the minmax principle to the estimate for matrix valued potentials we get

$$\sum_j (\lambda_j - \alpha)^{1/2} \leq \frac{1}{2} \int \text{Tr}[U(x) - \alpha]_+ \, dx$$

and inserting this in formula (1) leads to

$$\sum_j \lambda_j^\gamma \leq \frac{1}{\sqrt{\pi}} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 3/2)} \int \text{Tr}[U(x)]^{\gamma + 1/2} \, dx$$ \hspace{1cm} (2)

Using the matrix valued estimate for the $1/2$ powers yields

$$\text{Tr}[-\Delta - U]^{1/2} \leq \frac{1}{2} \int \text{Tr}[-\Delta' - U(x_1, x')]_- \, dx_1$$

Using (2) with $\gamma = 1$ yields

$$\text{Tr}[-\Delta - U]^{1/2} \leq \frac{1}{2} \frac{\Gamma(2)}{\Gamma(5/2)} \int dx_1 dx_2 \text{Tr}[-\partial^2_3 - U(x_1, x_2, x_3)]_-^{3/2}.$$ 

Now we use the sharp estimate for $\gamma = 3/2$ and get

$$\text{Tr}[-\Delta - U]^{1/2} \leq \frac{1}{2} \frac{1}{\sqrt{\pi}} \frac{\Gamma(2)}{\Gamma(5/2)} \frac{3}{16} \int U(x)^2 \, dx = \frac{1}{8\pi} \int U(x)^2 \, dx.$$ 

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To estimate \( \text{Tr}[-\Delta - U]_- \), we use (1) with \( \gamma = 1 \) to obtain

\[
\text{Tr}[-\Delta - U]_- \leq \frac{1}{\sqrt{\pi}} \frac{\Gamma(2)}{\Gamma(5/2)} \int dx_1 \text{Tr}[-\Delta' - U(x_1, x')]^{3/2}_-
\]

and applying the sharp estimate for the powers once for \( \gamma = 3/2 \) and then for \( \gamma = 2 \) yields

\[
\text{Tr}[-\Delta - U] \leq \frac{1}{\sqrt{\pi}} \frac{\Gamma(2)}{\Gamma(5/2)} \frac{3}{16} \int \int dx_1 dx_2 \text{Tr}[-\partial_3^2 - U(x_1, x_2, x')]^2_-
\]

\[
\leq \frac{1}{\sqrt{\pi}} \frac{\Gamma(2)}{\Gamma(5/2)} \frac{3}{16} \frac{8}{15\pi} \int U(x)^{5/2}_dx = \frac{2}{15\pi^2} \int U(x)^{5/2}_dx .
\]