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Stochastic basins of attraction for metastable states

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Basin of attraction of a stable equilibrium point is an effective concept for stability analysis in deterministic systems; however, it does not contain information on the external perturbations that may affect it. Here we introduce the concept of stochastic basin of attraction (SBA) by incorporating a suitable probabilistic notion of basin. We define criteria for the size of the SBA based on the escape probability, which is one of the deterministic quantities that carry dynamical information and can be used to quantify dynamical behavior of the corresponding stochastic basin of attraction. SBA is an efficient tool to describe the metastable phenomena complementing the known exit time, escape probability, or relaxation time. Moreover, the geometric structure of SBA gives additional insight into the system’s dynamical behavior, which is important for theoretical and practical reasons. This concept can be used not only in models with small noise intensity but also with noise whose amplitude is proportional or in general is a function of an order parameter. As an application of our main results, we analyze a three potential well system perturbed by two types of noise: Brownian motion and non-Gaussian \( \alpha \)-stable Lévy motion. Our main conclusions are that the thermal fluctuations stabilize the metastable system with an asymmetric three-well potential but have the opposite effect for a symmetric one. For Lévy noise with larger jumps and lower jump frequencies (\( \alpha = 0.5 \)) metastability is enhanced for both symmetric and asymmetric potentials. Published by AIP Publishing. [http://dx.doi.org/10.1063/1.4959146]

I. INTRODUCTION

Metastability is a universal phenomenon appearing in nature, with examples coming from physics: supersaturated vapours, magnetic systems with magnetization opposite to the external field; chemistry:16 chemical substances, reacting system; economics: crashes of financial markets, decisional system; biology: folding of proteins, enzyme and living organism; and climatology:2 effects of global warming; etc.

Analyzing the corresponding literature we could, however, not find a universal definition for metastability; it varies according to the developed approaches to deal with this issue. One of the examples of this notion and in our opinion the most general one was presented in Ref. 4. Here, by metastability is understood the phenomenon where a system, under the influence of a stochastic dynamics, explores its state space on different time scales: on fast ones transitions happen within a single subregion and on slow ones they occur between different subregions. The first attempt to formulate a rigorous dynamical theory of metastability was provided on physical grounds (see Ref. 20 and references therein). The first mathematical model for metastability was proposed in 1940 by Kramers and consists of diffusions in potential wells. Kramers formulated the average transition time and proved the exponential law. The subsequent research that contributed significantly to the development of
the mathematics of metastability were the work of Penrose and Lebowitz on metastable states in the van der Waals theory\textsuperscript{21} and the monograph of Freidlin and Wentzell on randomly perturbed dynamical systems\textsuperscript{9} where the large deviation theory for perturbed trajectories is established.

Two approaches to metastability are central within mathematics: i) the pathwise approach, as was introduced in Ref. 5, and is based on processing empirical averages along typical trajectories of the dynamics, in the spirit of the Freidlin–Wentzell theory. ii) The potential-theoretical approach which is based on an electric network perspective of the dynamics, focussing on crossing times via estimates on capacities.\textsuperscript{3} Presently, metastability is a highly active subfield of probability theory and statistical physics. For a recent overview there is a large body of literature on deterministic dynamical systems perturbed by random noise, where small Gaussian perturbations have been studied most extensively. The main references on this subject are Refs. 4, 20, and 24.

However, recently non-Gaussian Lévy motions are also often used in describing fluctuations in mathematical modeling of complex systems under uncertainty. As such phenomena can be considered: asset prices\textsuperscript{17}, a class of biological evolution\textsuperscript{27}, random search, Lagrangian drifts in certain oceanic fluid flows\textsuperscript{28}, climate, etc. One of the first attempts to support radical climate changes and build a model able to reproduce the effect of extreme events on climatic behavior was presented in Ref. 6. Commonly used Gaussian noise was replaced by \( \alpha \)-stable perturbations to model the stochastic forcing that atmospheric flow has on the oceanic flow.

The study of gradient dynamical systems subject to small perturbation by a heavy-tail Lévy process on a mathematical level of rigor was tackled in Ref. 13 for symmetric stable processes, where results on the first exit time from the potential well were established by purely probabilistic methods. It was shown that the exit time increases as a power of the small noise parameter and does not depend on the depth of the potential well but rather on the distance between the local minimum and the domain’s boundary.

In Refs. 10, 22, and 23 the mean first exit time and escape probability are utilized to quantify dynamical behaviors of stochastic differential equations (SDEs) with non-Gaussian \( \alpha \)-stable type Lévy motions. These deterministic quantities are characterized as solutions of the Balayage–Dirichlet problems of certain partial differential-integral equations which are hard to solve explicitly. In Ref. 22, asymptotic methods are offered to solve these equations to obtain escape probabilities when noises are sufficiently small. An efficient and accurate numerical scheme was developed and validated in Ref. 10 for computing the mean exit time and escape probability from the governing integro-differential equation.

So far two deterministic quantities have been considered—the mean exit time and the escape probability—to quantify the dynamical behavior of a stochastic differential equation (SDE)’s modeling metastability, especially, to describe how stable are the various states relative to each other. Metastability is a subtle phenomenon, with only a few known methods for its analysis, and so advances in this area are of fundamental importance. This demand motivates our research to provide powerful geometric tool for solving some applied problems. One of the attempts to provide a measure related to the volume of the basin of attraction for quantifying how stable a state is in a deterministic dynamical system in terms of basin stability was presented in Ref. 19. There the volume of the basin is regarded as an expression of the likelihood of return to a state after random—even non-small—perturbations. It is, however, a challenging open problem to generalize this concept of basin stability, which is a property of deterministic systems to the case of stochastically perturbed systems by introducing the notion of randomness in that tool.

To treat this problem, we propose here the concept of stochastic basin of attraction (SBA); in particular, we define a criterion for the size of the stochastic basin based on the escape probability. This concept can be used not only in models with small noise intensity but also in models with noise whose amplitude is proportional or even is a function of the order parameter, i.e., of the system state. We use the notion SBA for a basin of a metastable state in the sense that the criteria for the size of the basin is quantified by the escape probability, since the trajectory that starts in the domain of a metastable state remains there not forever but for a finite time only.

The efficiency of our SBA concept is shown by performing a comparative analysis between the method proposed in Ref. 25 and the SBA measure for the description of a relaxation phenomenon that in many natural systems proceed through metastable states and often are observed in condensed matter physics, cosmology, biology, and other fields. Studying the role of a thermal (\( T \)) and a nonequilibrium (\( \mu \)) Gaussian noise sources on the life time of the metastable state in a classical system with an asymmetric bistable potential the phenomenon of noise enhanced stability was revealed.\textsuperscript{8} The life time of the metastable state (that is the measure which was used in Ref. 8 to quantify the stability of the state) was calculated using the mean escape time only for a single initial condition. Thus it was determined that the optimum range of both types of noise intensities is \( T = 0.16 \) and \( \mu = 0.03 \), but so that the orbit’s escape probability from

\[ \text{FIG. 1. Phenomenon of noise enhanced stability by stochastic basin of attraction analysis.} \]
the metastable domain manifests high values which makes the former results on stabilization questionable. So we suggest here to complete this with the analysis of the size of SBA (Fig. 1). Due to the high escape probability, the size of the SBA of the metastable state with Gaussian noise is then significantly smaller than the SBA with Lévy noise, which recommends Lévy noise as the better candidate for the enhancement of metastability.

This approach will be specified in the following. We start with the assumptions about deterministic system and random perturbation (Sections II A and II B) and then define SBA for Gaussian and Lévy noise (Sections II C and II D), discuss an application (Sec. III), and finally give our conclusions (Sec. IV).

II. OBJECTIVES OF STUDY AND MAIN RESULTS

A. Deterministic dynamics

We consider an autonomous ordinary differential equation in the state space \( \mathbb{R}^n \)

\[
\dot{x} = f(x), \quad x(0; z) = z \in \mathbb{R}^n,
\]

where \( f \in C^2(\mathbb{R}^n, \mathbb{R}^n) \) is a vector field that generates a global solution flow \( \varphi_t(z) = x(t; z), z \in \mathbb{R}^n, \ t \geq 0 \) and satisfies a dissipativity condition (p. 128 in Ref. 9) which ensures the existence of a finite invariant measure of the trajectories of the perturbed system (Equation (5)):

There exist \( a, b > 0 \) such that \( \langle x, f(x) \rangle \leq a - b |x|^2 \), \( \forall x \in \mathbb{R}^n \), where \( \langle \cdot, \cdot \rangle \) and \(| \cdot |\) are the scalar product and the norm in \( \mathbb{R}^n \).

Let this dynamical system satisfy the following properties:

1. The set of nonwandering points of \( \varphi \) (see p. 236 in Ref. 11) consists of finitely many (local) attractors \( K_j \) for \( j = 1, \ldots, \kappa \) and \( \kappa \geq 1 \) (see this concept in Ref. 18), with their corresponding open domains of attractions \( D_j \).

2. All nonwandering points \( \xi_0 \) of \( \varphi \) are hyperbolic, i.e., the eigenvalues of \( Df(\xi_0) \) have a nonzero real part. We also assume that the family of eigenvectors of \( Df(\xi_0) \) is linearly independent. This assumption provides the structural stability (Theorem 1.8.3 in Ref. 8).

3. For each local attractor \( K_j \) there is a unique measure \( \mu_j \) supported on \( K_j \) (see Theorem 5.8.1, p. 282 in Ref. 11) with the following property: If \( U_j \) is a neighborhood of \( K_j \) such that \( \varphi_t(U_j) \subset U_j \) for all \( t \geq t_0 \), \( K_j = \cap_{t \geq t_0} \varphi_t(U_j) \), and \( g \) is a continuous function, then for almost all \( z \in U_j \) (with respect to Lebesgue measure)

\[
\lim_{t \to \infty} \frac{1}{T} \int_0^T g(\varphi_t(z))dt = \int_{K_j} g d\mu_j.
\]

Gradient systems with the vector field \( f(x) = -\nabla U(x) \) and finitely many minima \( s_j, j = 1, \ldots, \kappa \) are the simplest class of dynamical systems satisfying the above mentioned assumptions. In this case the potential \( U(x) \) increases faster than some linear function \( ax + b \) with increasing \( |x| \) and the local invariant measures are Dirac point-mass \( \mu_j = \delta_{s_j} \).

B. Random perturbations

Next we include random perturbations on system (1). Let us discuss just \( n \)-dimensional symmetric \( \sigma \)-stable Lévy motion; however, the definition of SBA given in Sec. II D is also applicable to the general case of Lévy motion with the generating triplet \((b, \sigma, \nu)\). For more details see Refs. 1 and 7.

For \( x \in [0, 2]\), an \( n \)-dimensional symmetric \( \sigma \)-stable motion is a Lévy motion \( L_t^\sigma \) with the characteristic function

\[
E e^{i(y \cdot L_t^\sigma)} = e^{-C\|y\|^2}, \quad y \in \mathbb{R}^n,
\]

where \( C = \pi^{-(n/2)} \Gamma((n+2)/2) \Gamma((n+4)/2) / \Gamma((n+2)/2) \). Note that \( C = 1 \) for the dimension \( n = 1 \). A symmetric 2-stable process is simply a Brownian motion. The generating triplet of the symmetric \( \sigma \)-stable process \( L_t^\sigma \) is \((0, 0, \sigma)\), with the jump measure

\[
\nu_\sigma(dy) = c(n, \sigma) \frac{dy}{\|y\|^n}, \quad x \in [0, 2),
\]

where \( c(n, \sigma) \) is the intensity constant and \( \| \cdot \| \) is the usual Euclidean norm in \( \mathbb{R}^n \).

We now perturb the deterministic dynamical system (1) by a non-Gaussian Lévy motion as well as by a Brownian motion obtaining the following SDE in \( \mathbb{R}^n \)

\[
dX_t = f(\xi_t)dt + \sigma(\xi_t)dB_t + dL_t^\sigma, \quad X_0 = x,
\]

where \( B_t \) is a standard Brownian motion in \( \mathbb{R}^n \), \( L_t^\sigma \) is a symmetric \( \sigma \)-stable Lévy motion in \( \mathbb{R}^n \), \( f \) is a vector-valued function, and \( \sigma \) is an \( n \times n \) matrix-valued function.

We make some appropriate assumptions on the vector field \( f \), the noise intensity \( \sigma \), and the jump measure \( \nu \); so that SDE (5) has a unique, adapted, cadlag solution as follows.\(^1\) In the following, \( \| \sigma \|^2 = \sum_{j=1}^n \sum_{i=1}^n \sigma_{ij}^2 \) denotes a matrix norm.

1. Lipschitz condition: There exists a positive constant \( K_1 \) such that for all \( x_1, x_2 \in \mathbb{R}^n \)

\[
||f(x_1) - f(x_2)||^2 + \|\sigma(x_1) - \sigma(x_2)\|^2 \leq K_1 \|x_1 - x_2\|^2.
\]

2. Growth condition: There exists a positive constant \( K_2 \) such that for all \( x \in \mathbb{R}^n \)

\[
||f(x)||^2 + \|\sigma(x)\|^2 + \int_{|y| < 1} \|y\|^2 \nu(dy) \leq K_2 (1 + ||x||^2).
\]

The generator \( A \) in terms of the partial derivatives of \( g(x) \) for the solution process \( X_t \) is

\[
Ag(x) = \sum_{i=1}^n f_i(x) \frac{\partial g}{\partial x_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\sigma \sigma^T)_{ij} \frac{\partial^2 g}{\partial x_i \partial x_j}
\]

\[
+ \int_{\mathbb{R}^n \setminus \{0\}} \left[ g(x+y) - g(x) \right] \nu_\sigma(dy),
\]

where \( g(x) \in C^2(\mathbb{R}^n) \).
C. The SBA for the Gaussian case

In the following, we discuss Gaussian and non-Gaussian noise separately. Let us first consider system (5) without a Lévy term. Let $D$ be one of the open domains of attractions of one of the attractors $K$ defined in Sec. II A (1). By the assumption that we considered in Sec. II A for the deterministic dynamical system $\dot{u} = f(u)$, the boundary $\partial D$ of the domain $D$ contains portions of the boundary of the nearby invariant domains, i.e., $\partial D$ can be divided into $n$ (it depends on the number of nearby invariant domains) mutually exclusive subsets $\Gamma_j$. Define the first exit time, i.e., the time the trajectory first leaves $D$

$$\tau_D := \inf \{ t > 0 : X_t \notin D \}, \quad (9)$$

where $D^c$ is the complement of $D$ in $\mathbb{R}^n$. When $X_t$ has an almost surely continuous path, a solution orbit starting at $x \in D$ will hit $D^c$ by hitting $\partial D$ first. Thus, $\tau_D = \tau_{\partial D}$. The likelihood that $X_t$, starting at $x$, exits from $D$ first through $\Gamma_j$ is called the escape probability $p_i(x)$ from $D$ through $\Gamma_j$

$$p_i(x) = P\{ X_{\tau_D} \in \Gamma_j \}. \quad (10)$$

Next we define the stochastic basin of attraction (SBA) for an attractor $K$.

**Definition II C.1.** The stochastic basin of attraction $B_K(m,M)$ of the attractor $K$ in the dynamical system driven by Gaussian noise is the subset of the boundary $\partial D$ of the attractor $K$ which is the set of all initial conditions $X_0 = x$ whose solutions $X_t$ satisfy the following criteria:

Criterion I (that defines sets $D_d$ and $D_I = \bigcap_{i=1}^n D_d$ with smooth boundary $\partial D_I$): $p_i(x) < m$ ($m$ is the stability level I), where $p_i(x)$ is the escape probability from $D$ through $\Gamma_j$ ($p_i(x) = P\{ X_{\tau_D} \in \Gamma_j \}$);

Criterion II (that defines sets $D_{dI}$): $p_i(x) > M$ ($M$ is the stability level II), where $p_i(x)$ is the escape probability from $D$ through $\Gamma_j$ which is the subset of the boundary $\partial D_I$ ($p_i(x) = P\{ X_{\tau_D} \in \Gamma_{dI} \}$).

Roughly speaking the SBA is the set of initial conditions whose solutions have a “small” probability (measured by the level $m$ set out in criterion I) of exits from the neighborhood of the attractor $K$ and a “high” probability (measured by the level $M$ set out in criterion II) of returns to the vicinity of $K$. In the case when the attractor has more than one neighbor attractor, the set $D_I$ (defined by criterion I) may be interpreted as the almost equiprobable set escape. This is the set of initial conditions whose solutions have almost the same probability to escape from a domain through any part $\Gamma_j$ of the boundary $\partial D$ (see Fig. 2). The values for $m$ and $M$ have to be chosen appropriately.

In Theorem 5.19 of Ref. 7, it was shown that the escape probability $p_i(x)$ defined in (10) solves a linear elliptic partial differential equation, with the specifically chosen Dirichlet boundary condition and therefore can be calculated numerically

$$A p_i = 0,$$

$$p|_{\Gamma_j} = 1,$$

$$p|_{\partial D \cap \Gamma_j} = 0,$$ \quad (11)

in the case of criterion I and

$$Ap_i = 0,$$

$$p|_{\Gamma_j} = 1,$$

$$p|_{\partial D \cap \Gamma_j} = 0,$$ \quad (12)

in the case of criterion II, where $A$ is the generator for the stochastic system (5) in the Gaussian case

$$A g(x) = \sum_{i=1}^n f_i(x) \frac{\partial g}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma^T)_{ij} \frac{\partial^2 g}{\partial x_i \partial x_j}. \quad (13)$$

As the well-known concept of basin, the introduced SBA, representing the set of initial conditions in the state space whose orbits behave in a “similar way,” is the important geometric structure that can be used to understand and describe the metastable behavior of a system. Thus, it is important to have resources for describing the basin of attraction and quantifying its shape and size for theoretical and practical reasons.26

D. The SBA for the non-Gaussian Lévy case

For a dynamical system driven by a Lévy motion, almost all the orbits $X_t$ have jumps in time. In fact, these orbits are càdlàg (right-continuous with left limit at each time instant), i.e., each of these orbits has at most countably many jumps on any finite time interval. Due to these jumps, an orbit could escape a domain without passing through its boundary. So the definition of an SBA presented above has to be generalized to this case.

In this section, we will consider the SDE (5) driven by Brownian as well as Lévy motions. As in Sec. II C, $D$ is the open domain of attraction of an attractor $K$. By the assumption that we considered in Sec. II A for the deterministic dynamical system $\dot{u} = f(u)$, the $D^c$ space is divided by separating manifolds in the $D_I$ domains of attraction of nearby attractors $K_j$.

We define the first exit time analogous to (10). When $X_t$ has càdlàg paths which have countable jumps in time, the first hitting of $D^c$ may occur either on the boundary $\partial D$ or already somewhere in $D^c$. For this reason, we take a subset $D'_j$ of $D^c$, and define the likelihood that $X_t$ exit firstly from $D$ by landing in the target set $D'_j$ as the escape probability $p_i(x)$ from $D$ to $D'_j$

$$p_i(x) = P\{ X_{\tau_D} \in D'_j \}. \quad (14)$$
We define now the stochastic basin of attraction for an attractor $K$. 

**Definition IID.1.** The stochastic basin of attraction $B_K(m,M) = [\bigcup_{i=1}^{n} D_{c_i}^{II}] \cup D_I$ of the attractor $K$ is the set of all initial conditions $X_0 = x$ whose solutions $X_t$ satisfy the following criteria:

**Criterion I** (that defines the sets $D_{c_i}^{II}$ and $D_I$):

$p_i(x) < m$ (m is the stability level I), where $p_i(x)$ is the escape probability from $D$ to $D_{c_i}^{II}$ ($p_i(x) = P\{X_{t_{cg}} \in D_{c_i}^{II}\}$).

**Criterion II** (that defines the sets $D_{c_i}^{II}$):

$p_i(x) > M$ (M is the stability level II), where $p_i(x)$ is the escape probability from $D_{c_i}^{II}$ to $D_I$ ($p_i(x) = P\{X_{t_{cg}} \in D_I\}$).

A graphical interpretation of the SBA is represented in Fig. 3.

In Theorem 7.35 of Ref. 7, it was shown that the escape probability $p_i(x)$ defined in (14) solves a differential-integral equation, with the specifically chosen Dirichlet boundary conditions and also can be calculated numerically:

\[
A p_i = 0, \\
p_i|_{D_I} = 1, \\
p_i|_{D_{c_i}^{II}} = 0.
\]  

in the case of criterion I and

\[
A p_i = 0, \\
p_i|_{D_I} = 1, \\
p_i|_{(D_{c_i}^{II}) \setminus D_I} = 0,
\]

in the case of criterion II, where $A$ is the generator (8) for the stochastic system (5).

### III. APPLICATION

The goal of this section is on the one hand to operationalize the computation of the new SBA concept and on the other hand to demonstrate the practical benefit of SBA in quantification of influence of noise on metastability. As the escape probability is described by the differential-integral equations (15) and (16), we use a numerical approach adapted from Ref. 10 for solving it. A computational analysis is conducted to investigate the importance of the deterministic vector field $f$, the type of noise, the strength of the thermal, and intensity of the nonequilibrium fluctuations in the case of purely Gaussian noise as well as the importance of the jump measure, the diffusion coefficient, and non-Gaussianity in affecting the size and location of the SBA and consequently metastability of the states.

We consider the stochastic dynamical system (5) in $\mathbb{R}^1$ with the deterministic vector field $f(x) = -\nabla U$ to study the impact of noise on the dynamics of the systems with different geometry of the vector field. The SBA is an appropriate measure to quantify this effect and perform a comparative analysis. For this propose we will consider two types of well potential, first

\[
U(x) = \frac{1}{6}x^6 - \frac{5}{4}x^4 + 2x^2
\]

is a symmetric three-well potential, with three minima at $x = -2$, $x = 0$, and $x = 2$ (metastable states), and two maxima at $x = -1$ and $x = 1$ (see Figs. 4(a) and 5(a)), and second
is an asymmetric three-well potential, with three minima at \(x = -2, x = -0.5,\) and \(x = 2\) (metastable states), and two maxima at \(x = -1\) and \(x = 1\) (see Figs. 4(b) and 5(b)).

It is known that the geometry of the deterministic vector field plays a crucial role in both theories describing metastable phenomena in the purely probabilistic approach by Freidlin and Wentzell for induced transitions among the metastable states and quantify noise fluctuation, which is often used in physics, engineering, economics, ecology, and other areas to describe noise-induced transitions among the metastable states and quantify the absorbing states. The form of the functional dependence is considered according to the studied phenomenon and varies from a pure linear to more complex forms. In this case our main results are the following:

The negligible noise effect on the stability of the state \(K = 2\) (in both potentials) and of the state \(K = -2\) (in the symmetric one). The SBA of the attractor \(K = 2, B_2(0.6, 0.7): (0.96, +\infty), (0.96, +\infty), (1, +\infty),\) and \((0.96, +\infty)\) is basically represented by the \(D_I\) set, defined in criterion \(I,\) and does not change so much with alterations in the intensities of both \(T\) and \(\mu\) noise parameters. The SBA of \(K = 2\) does not differ from deterministic basin indicating that the noise has no significant effect on the state stability.

In the symmetric potential the SBA of the metastable state \(K = -2, B_{-2}(0.6, 0.7): (-\infty, -0.8), (-\infty, -0.8), (-\infty, -0.9), (-\infty, -0.9)\) (for the attractor \(K = 2\) the SBA are the symmetric sets to those) does not differ too much for the different levels of \(T\) and \(\mu\) parameters (Fig. 4(a)) and also from deterministic basin which indicates stability resistance against perturbations. This is due to the notable height of the potential barrier that the process must overcome to escape from the domain \((-\infty, -1)\) or \((1, +\infty)\). The calculations of the SBA of the metastable states \(K = 0\) and \(K = -2\) are shown in Fig. 6.

**The noise induced absorbing state \(K = -2\) and vanishing of \(K = -0.5\)'s basin.** Due to the significant difference among the sizes of the SBA of the three metastable states (Fig. 4(b)) we can conclude that the deepest well \(K = -2\) of the asymmetric potential is the absorbing one, including not only the deterministic basin of the attractor \(K = -2\) but also a predominant part of the neighbor attractor \(K = -0.5\). To this increase in SBA of the attractor \(K = -2\) a remarkable contribution refers to the sets \(D^0_I: (-1.28, 0.41), (-1.22, 0.6), (-1.26, 0.57), (-1.12, 0.81)\) which, according to criterion \(I\) include all initial conditions \(X_0 = x\) whose solutions \(X_t\) have the escape probability from \(D^0_I: p(x) > 0.7\). For models with a small thermal fluctuation intensity the length of the SBA of the metastable state \(K = -0.5\) is almost non-significant 0.17 (for \(T = 0.05, \mu = 0\)) and 0.3 (for \(T = 0.05, \mu = 0.45\)), including initial points of equiprobable escape \(X_0 = 0.88\) and \(X_0 = 0.82\), respectively (Fig. 4(b)). The reduced size of the SBA of the attractor \(K = -0.5\) is directly related with the high escape probability of the solutions through \(\Gamma_1 = -1\) and therefore with low escape probability through \(\Gamma_1 = 1\), due to the height of the right potential well (Fig. 7(b)).

**The thermal fluctuation stabilizes the metastable system with an asymmetric three-well potential but has the opposite effect for a symmetric one.** We especially find that the length of the SBA of the metastable state \(K = 0\) in symmetric well potential for the four different combinations of \(T\) and \(\mu\) is roughly the same \(B_0(0.6, 0.7): (-0.83, 0.83), (-0.8, 0.8), (-0.88, 0.88),\) and \((-0.91, 0.91)\) (Figs. 4(a) and 8). However, the contribution of each of the criteria (given in Def. IIC.1) for this basin is different. So the size of the sets \(D_I: (-0.34, 0.34), (-0.36, 0.36), (-0.5, 0.5),\) and \((-0.68, 0.68)\), which is defined in criterion \(I,\) is reducing with the increase of the \(T\) and

\[
U(x) = \frac{1}{6}x^6 + \frac{1}{10}x^5 - \frac{5}{4}x^4 - \frac{5}{6}x^3 + 2x^2 + 2x
\]
noise parameters, since it is increasing the probability of escape (Fig. 6(a)). This reduction is more significant with the increase of the thermal noise parameter, maintaining the level of the nonequilibrium noise constant. The length of the intervals $D_{c1}$ and $D_{c2}$, defined in criterion II, increases as the intensity of the noises grows, thus contributing to the equality of SBA of the metastable state $K = 0$ for the different levels of the $\mu$ and $T$ noise parameters. On the other hand, the length of SBA of the metastable state $K = -0.5$ in asymmetric well potential increases from 0.17 to 0.37 (for $\mu = 0$) and from 0.3 to 0.43 (for $\mu = 0.45$).

**B. Three-well potential perturbed by $\alpha$-stable Lévy noise**

For the Lévy noise equation (5) is specified to

$$dX_t = (-\nabla U)dt + \sqrt{d} dB_t + dL_t^{\alpha}, \ X_0 = x, \quad (20)$$

where the variance (or diffusion) $d > 0$.

Here we mainly find that for Lévy noise with larger jumps and lower jump frequencies ($\alpha=0.5$) metastability is enhanced for both symmetric and asymmetric potentials. The size of the set $D_I$ for $K = 0$ and $K = -0.5$ defined by criterion I (in Def. IID.1 we call it almost equiprobable set escape) for smaller values of $\alpha$ ($0 < \alpha < 1$) represents higher values, due to the flatness of the escape probability away from the boundary of the domain (see Figs. 9(b) and 10(a)). This implies that the jump sizes of the process are usually larger than the domain size, and thus the escape probability has small variations for different starting positions. Due to the same reason for smaller values of $\alpha$, the intervals $D_{cII}$ defined by criterion II do not contribute so much to the increase in SBA.

We observed the similar noise induced metastable effect for Gaussian ($T=45, \mu=0$) and Lévy...
(α=1.5, d=0) perturbations. The similarities are found in both: in the size of the basins (see Fig. 8) as well as in the particular characteristics of metastability. Among these are: noise induced absorbing state $K = -2$, vanishing basin for the attractor $K = -0.5$ in the case of the asymmetric potential and the strong noise effect on the $K = 0$, reduced influence on the $K = -2, K = 2$. One of the possible reasons of these resemblances is the proximity of the $α$-parameter to the value 2 when Levy noise assumes Gaussian characteristics.

<table>
<thead>
<tr>
<th>Noise type</th>
<th>Parameters</th>
<th>Symmetric well potential (metastable states)</th>
<th>Asymmetric well potential (metastable states)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>K=2</td>
<td>K=2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$B_2(0.6,0.7)$</td>
<td>$B_2(0.6,0.7)$</td>
</tr>
<tr>
<td>Gaussian</td>
<td></td>
<td>$D_1$</td>
<td>$D_1$</td>
</tr>
<tr>
<td>$α=0.45, μ=0.45$</td>
<td>$(-0.80,1.23)$</td>
<td>$(-0.83,0.34)$</td>
<td>$(0.85,\infty)$</td>
</tr>
<tr>
<td>$α=0.45, μ=0$</td>
<td>$(-0.80,1.17)$</td>
<td>$(-0.80,0.36)$</td>
<td>$(0.85,\infty)$</td>
</tr>
<tr>
<td>$α=0.05, μ=0.45$</td>
<td>$(-0.90,1.21)$</td>
<td>$(-0.88,0.88)$</td>
<td>$(0.90,\infty)$</td>
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<tr>
<td>$α=0.05, μ=0$</td>
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<tr>
<td>$α=0.5, d=0$</td>
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<td>$(-1,1)$</td>
<td>$(1,\infty)$</td>
</tr>
<tr>
<td>$α=1.5, d=2$</td>
<td>$(-0.5)$</td>
<td>$(-0.7,0.7)$</td>
<td>$(0.5,\infty)$</td>
</tr>
<tr>
<td>$α=1.5, d=0$</td>
<td>$(-0.7)$</td>
<td>$(-0.8,0.8)$</td>
<td>$(0.7,\infty)$</td>
</tr>
</tbody>
</table>

FIG. 7. Gaussian case. The SBA of the metastable state $K = -0.5$, dashed black line ($T = 0.45, μ = 0.45$), dashed blue line ($T = 0.45, μ = 0$), solid black line ($T = 0.05, μ = 0.45$), and solid blue line ($T = 0.05, μ = 0$). (a) Set $D_{III}$ defined by Criterion II. Escape probability from $D_1 = (-2.1,0.68), (-2.1,0.7), (-2.1,0.78), (-2.1,0.86)$, respectively, through $Γ_1$. (b) Set $D_2$ defined by Criterion I. Escape probability from $D = (-1,1)$ through $Γ_1 = -1$ and $Γ_2 = 1$. (c) Set $D_{III}$ defined by Criterion II. Escape probability from $D_2 = (0.8,2.1), (0.8,2.1), (0.86,2.1), (0.92,2.1)$, respectively, through $Γ_2$.

FIG. 8. The SBA of the metastable states in the symmetric and asymmetric three-well potential perturbed by Gaussian and $α$-stable Lévy noise.
In the asymmetric three-well potential the equiprobable escape starting point ($X_0^{5}:X_0^{8}:X_0^{9}$ for respective values of $a$ and $d$) is far away from the stable point of the deterministic system in the direction of the highest and farther potential barrier. In the purely probabilistic approach for diffusions with a small $L_c^1$levy noise intensity, this remoteness is due to the distance between the local minimum and the domain’s boundary as well as in the classical large deviations theory this is due to the height of the potential barrier that trajectories have to overcome.

In the case of the symmetric three-well potential, due to the symmetry of the domains and the escape probability, the SBA of the metastable states is symmetric with respect to the y-axis and the equiprobable escape starting point coincides with the deterministic attractor $K = 0$.

**IV. CONCLUSION**

We have introduced here a new measure related to the volume of the SBA of a metastable state in a stochastic dynamical system with two types of perturbations: Brownian motion and non-Gaussian $\alpha$-stable type Lévy motion. We have defined criteria for the size of the SBA based on the escape probability, which is one of the deterministic quantities that carry dynamical information of the SDE.

Roughly speaking, the SBA is the set of initial conditions whose solutions have a “small” probability (measured by the level $m$ set out in criterion I) of exits from a neighborhood of the attractor $K$ and a “high” probability (measured by the level $M$ set out in criterion II) of returns to the vicinity of $K$. In the case when the attractor has more than one neighboring attractor, the set $D_I$ (defined by criterion I) may be interpreted as the almost equiprobable set escape. This is the set of initial conditions whose solutions have almost the same probability to escape from a domain $D$ through any part $\Gamma_I$ of the boundary $\partial D$ or by landing in any target set $D^c_I$ of $D^c$.

Furthermore, we have also presented a numerical method to conduct the computation of the new SBA concept. As an application of our main result, we consider a dynamical system

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**FIG. 9.** Non-Gaussian case. The SBA of the metastable state $K = -0.5$, dotted line ($a = 0.5, d = 0$), solid line ($a = 1.5, d = 0$), and dashed line ($a = 1.5, d = 2$). (a) Set $D_{II}^c$ defined by Criterion II. Escape probability from $D_{II} = (-2.1, 0.32), (-2.1, 0.7), (-2.1, 0.17)$, respectively, to $D_I$. (b) Set $D_I$ defined by Criterion I. Escape probability from $D = (-1, 1)$ to $D_I^c = (-\infty, -1)$ and $D_I^c = (1, +\infty)$. (c) Set $D_{II}^c$ defined by Criterion II. Escape probability from $D_{II}^c = (0.65, 2.1), (0.86, 2.1), (1.21, 1.21)$, respectively, to $D_I$. 

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in $\mathbb{R}^1$ with a vector field $-\nabla U$, where $U$ is a symmetric as well as an asymmetric three-well potential perturbed by two types of noises, Brownian motion and $\alpha$-stable Lévy motion.

A computational analysis is conducted to investigate the importance of the deterministic vector field, the type of noise, the strength of the thermal, and intensity of the nonequilibrium fluctuations in the case of purely Gaussian noise as well as the importance of the jump measure, the diffusion coefficient, and non-Gaussianity in affecting the size and location of the SBA and consequently metastability of the states.

Our main conclusion is that the thermal fluctuation stabilizes the metastable system with an asymmetric three-well potential but has the opposite effect for a symmetric one. On the one hand we have observed the negligible noise effect on the stability of the state $K=2$ (in both potentials) and of the state $K=-2$ (in the symmetric one); on the other hand the analysis of SBA reveals the noise induced absorbing state $K=-2$ and vanishing $K=-0.5$’s basin. We conclude also that for Lévy noise with larger jumps and lower jump frequencies ($\alpha=0.5$) metastability is enhanced for both symmetric and asymmetric potentials. According to the obtained results the similar noise induced metastable effect was observed for Gaussian ($T=45, \mu=0$) and Lévy ($\alpha=1.5, d=0$) perturbations.

As the well-known concept of basin, the introduced SBA, representing the set of initial conditions in the state space whose orbits behave in a “similar way,” is the important geometric structure that can be used to understand and describe the metastable behavior of a system. Thus, we suggest to include such a calculation and analysis of the SBA as part of the routine investigation of the metastable state for any stochastic dynamical system describing metastable phenomena.

FIG. 10. Non-Gaussian case. The SBA of the metastable state $K=0$ and $K=-2$, dotted line ($\alpha=0.5, d=0$), solid line ($\alpha=1.5, d=0$), and dashed line ($\alpha=1.5, d=2$). (a) $K=0$, set $D_I$ defined by Criterion I. Escape probability from $D=(-1,1)$ to $D'_I=(-\infty,-1)$ and $D''_I=(1,\infty)$. (b) $K=0$, set $D'_I$ defined by Criterion II. Escape probability from $D'_I=(-2.1,-0.9), (-2.1,-0.5), (-2.1,-0.3)$, respectively, to $D_I=(-0.9,0.9), (-0.5,0.5), (-0.3,0.3)$, respectively. (c) $K=-2$, set $D_I$ defined by Criterion I. Escape probability from $D=(-2.1,-1)$ to $D'=(1,\infty)$. (d) $K=-2$, set $D'_I$ defined by Criterion II. Escape probability from $D'=(1,2.1), (1.1,2.1), (1.3,2.1)$, respectively, to $D_I=(-\infty,-1), (-\infty,-1.1), (-\infty,-1.3)$, respectively.

