Asymptotic Zero Distribution of Biorthogonal Polynomials

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Abstract

Let \( \psi : [0,1] \rightarrow \mathbb{R} \) be a strictly increasing continuous function. Let \( P_n \) be a polynomial of degree \( n \) determined by the biorthogonality conditions

\[
\int_0^1 P_n(x) \psi(x)^j \, dx = \begin{cases} 
0, & j = 0, 1, \ldots, n-1, \\
I_n \neq 0, & j = n
\end{cases}
\]  

We study the distribution of zeros of \( P_n \) as \( n \to \infty \), and related potential theory.

1. Introduction and Results

Let \( \psi : [0,1] \rightarrow [\psi(0), \psi(1)] \) be a strictly increasing continuous function, with inverse \( \psi^{-1} \). Then we may uniquely determine a monic polynomial \( P_n \) of degree \( n \) by the biorthogonality conditions

\[
\int_0^1 P_n(x) \psi(x)^j \, dx = \begin{cases} 
0, & j = 0, 1, 2, \ldots, n-1, \\
I_n \neq 0, & j = n
\end{cases}
\]  

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$P_n$ will have $n$ simple zeros in $(0,1)$, so we may write

$$P_n(x) = \prod_{j=1}^{n} (x - x_{jn}).$$

The proof of this is the same as for classical orthogonal polynomials. Our goal in this paper is to investigate the zero distribution of $P_n$ as $n \to \infty$. Accordingly, we define the zero counting measures

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{x_{jn}},$$

that place mass $\frac{1}{n}$ at each of the zeros of $P_n$, and want to describe the weak limit(s) of $\mu_n$ as $n \to \infty$.

This topic was initiated by the second author, in the course of his investigations on convergence acceleration \[8\], \[24\], and numerical integration of singular integrands. He considered \[21\], \[22\], \[23\]

$$\psi(x) = \log x, \quad x \in (0,1)$$

and found that the corresponding biorthogonal polynomials are

$$P_n(x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \left( \frac{j + 1}{n + 1} \right)^j x^j.$$ 

The latter are now often called the *Sidi polynomials*, and one may represent them as a contour integral. Using steepest descent, the strong asymptotics of $P_n$, and their zero distribution, were established in \[14\]. Asymptotics for more general polynomials of this type were analyzed by Elbert \[7\]. Extensions, asymptotics, and applications in numerical integration, and convergence acceleration have been considered in \[15\], \[16\], \[25\], \[26\]. Biorthogonal polynomials of a more general form have been studied in several contexts – see \[5\], \[10\], \[11\]. The sorts of biorthogonal polynomials used in random matrices \[3\], \[6\], \[12\] are mostly different, although there are some common ideas in the associated potential theory.

Herbert Stahl’s interest in this topic arose after he refereed \[14\]. He and the first author discussed the topic at some length at a conference in honor of Paul Erdős in 1995. This led to a draft paper on zero distribution in the later 1990’s, with revisions in 2001, and 2003, and this paper is the partial completion of that work. For the case $\psi(x) = x^{\alpha}, \quad \alpha > 0$, we presented
explicit formulae in [18]. Rodrigues type representations were studied in [17].

Distribution of zeros of polynomials is closely related to potential theory [1], [20], [28], and accordingly we introduce some potential theoretic concepts. We let \( \mathcal{P} (\mathcal{E}) \) denote the set of all probability measures with compact support contained in the set \( \mathcal{E} \). For any positive Borel measure \( \mu \), we define its classical energy integral

\[
I (\mu) = \int \int \log \frac{1}{|x-t|} \, d\mu (x) \, d\mu (t),
\]

and denote its support by \( \text{supp} [\mu] \). Where appropriate, we consider these concepts for signed measures too. For any set \( \mathcal{E} \) in the plane, its (inner) logarithmic capacity is

\[
\text{cap} (\mathcal{E}) = \sup \left\{ e^{-I (\mu)} : \mu \in \mathcal{P} (\mathcal{E}) \right\}.
\]

We say that a property holds q.e. (quasi-everywhere) if it holds outside a set of capacity 0. We use \( \text{meas} \) to denote linear Lebesgue measure 0. For further orientation on potential theory, see for example [13], [19], [20].

In our setting we need a new energy integral

\[
J (\mu) = \int \int K (x,t) \, d\mu (x) \, d\mu (t),
\]

where

\[
K (x,t) = \log \frac{1}{|x-t|} + \log \frac{1}{|\psi (x) - \psi (t)|}.
\]

In [6], a similar energy integral was considered for \( \psi (t) = e^t \), but with an external field. The minimal energy corresponding to \( \psi \) is

\[
J^* (\psi) = \inf \{ J (\mu) : \mu \in \mathcal{P} ([0,1]) \}.
\]

Under mild conditions on \( \psi \), we shall prove that there is a unique probability measure, which we denote by \( \nu_\psi \), attaining the minimum. For probability measures \( \mu, \nu \), we define the classical potential

\[
U^\mu (x) = \int \log \frac{1}{|x-t|} \, d\mu (t),
\]

the mixed potential

\[
W^{\mu,\nu} (x) = \int \log \frac{1}{|x-t|} \, d\mu (t) + \int \log \frac{1}{|\psi (x) - \psi (t)|} \, d\nu (t)
\]

\[= U^\mu (x) + U^{\nu_{\psi^{-1}} \circ \psi} (x), \]
and the $\psi$ potential

$$W^\mu (x) = W^{\mu,\mu} (x) = \int K (x, t) \, d\mu (t). \quad (11)$$

We note that potential theory for generalized kernels is an old topic, see for example, Chapter VI in [13]. However, there does not seem to be a comprehensive treatment covering our setting. Our most important restrictions on $\psi$ are contained in:

**Definition 1.1.** Let $\psi : [0, 1] \to [\psi (0), \psi (1)]$ be a strictly increasing continuous function, with inverse $\psi^{[-1]}$. Assume that $\psi$ satisfies the following two conditions:

(I) $\text{cap} (E) = 0 \Rightarrow \text{cap} (\psi^{[-1]} (E)) = 0. \quad (12)$

(II) For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{meas} (E) \leq \delta \Rightarrow \text{meas} (\psi^{[-1]} (E)) \leq \varepsilon. \quad (13)$$

Then we say that $\psi$ preserves smallness of sets.

The conditions (I), (II) are satisfied if $\psi$ satisfies a local lower Lipschitz condition. By this we mean that we can write $[0, 1]$ as a countable union of intervals $[a, b]$ such that in $[a, b]$, there exist $C, \alpha > 0$ depending on $a, b$, with

$$|\psi (x) - \psi (t)| \geq C |t - x|^\alpha, x, t \in [a, b].$$

We can apply Theorem 5.3.1 in [19, p. 137] to $\psi^{-1}$ to deduce (12).

Using classical methods, we shall prove:

**Theorem 1.2.** Let $\psi : [0, 1] \to [\psi (0), \psi (1)]$ be a strictly increasing continuous function that preserves smallness of sets. Define the minimal energy $J^* = J^* (\psi)$ by (7). Then

(a) $J^*$ is finite and there exists a unique probability measure $\nu_\psi$ on $[0, 1]$ such that

$$J (\nu_\psi) = J^*. \quad (14)$$

(b) $W^{\nu_\psi} \geq J^*$ q.e. in $[0, 1]. \quad (15)$

In particular, this is true at each point of continuity of $W^{\nu_\psi}$.
(c) \[ W^{\nu_\psi} \leq J^* \text{ in } \text{supp}[\nu_\psi]. \]  
\[ (16) \]
and \[ W^{\nu_\psi} = J^* \text{ q.e. in } \text{supp}[\nu_\psi]. \]  
\[ (17) \]

(d) \( \nu_\psi \) is absolutely continuous with respect to linear Lebesgue measure on \([0,1]\). Moreover, there are constants \( C_1 \) and \( C_2 \) depending only on \( \psi \), such that for all compact \( K \subset [0,1] \),
\[ \nu_\psi(K) \leq \frac{C_1}{|\text{log cap}K|} \leq \frac{C_2}{|\text{log meas}(K)|}. \]  
\[ (18) \]

(e) There exists \( \varepsilon > 0 \) such that \([0,\varepsilon] \cup [1-\varepsilon,1] \subset \text{supp}[\nu_\psi]\).  
\[ (19) \]

Let
\[ I_n = \int_0^1 P_n(t) \psi(t)^n \, dt, \quad n \geq 1. \]  
\[ (20) \]

**Theorem 1.3.** Let \( \psi : [0,1] \to [\psi(0),\psi(1)] \) be a strictly increasing continuous function that preserves smallness of sets. Let \( \{P_n\} \) be the corresponding biorthogonal polynomials, with zero counting measures \( \{\mu_n\} \). If \[ \text{supp}[\nu_\psi] = [0,1], \]  
\[ (21) \]
then the zero counting measures \( \{\mu_n\} \) of \( (P_n) \) satisfy
\[ \mu_n \rightharpoonup^* \nu_\psi, n \to \infty \]  
\[ (22) \]
and
\[ \lim_{n \to \infty} I_n^{1/n} = \exp(-J^*). \]  
\[ (23) \]

The weak convergence (22) is defined in the usual way:
\[ \lim_{n \to \infty} \int_0^1 f(t) \, d\mu_n(t) = \int_0^1 f(t) \, d\nu_\psi(t), \]
for every continuous function \( f : [0,1] \to \mathbb{R} \). We can replace (21) by the more implicit, but more general, assumption that \( \text{supp}[\nu_\psi] \) contains the support of every weak limit of every subsequence of \( (\mu_n) \). We can at least prove it when the kernel \( K \), and hence the potential \( W^{\nu_\psi} \), satisfies a convexity condition:
Theorem 1.4. Let $\psi : [0, 1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function that preserves smallness of sets. In addition assume that $\psi$ is twice continuously differentiable in $(0, 1)$ and either

(a) for $x, t \in (0, 1)$ with $x \neq t$,

$$\frac{\partial^2}{\partial x^2} K(x, t) > 0,$$  \hfill (24)

or

(b) for $x, t \in (\psi(0), \psi(1))$ with $x \neq t$,

$$\frac{\partial^2}{\partial x^2} \left[ K\left(\psi^{-1}(x), \psi^{-1}(t)\right)\right] > 0.$$  \hfill (25)

Then

$$\text{supp } [\nu_\psi] = [0, 1].$$  \hfill (26)

Example. Let $\alpha > 0$ and

$$\psi(x) = x^\alpha, \quad x \in [0, 1].$$

Then either (25) or (26) holds and hence (21) holds. We show this separately for $\alpha \geq 1$ and for $\alpha < 1$.

Case I $\alpha \geq 1$

We shall show that the hypotheses of Theorem 1.4 (a) are fulfilled. A straightforward calculation gives that

$$\Delta(x, t) := (x - t)^2 (\psi(x) - \psi(t))^2 \frac{\partial^2}{\partial x^2} K(x, t)$$

$$= (x^\alpha - t^\alpha)^2 + (\alpha x^{\alpha-1})^2 (x - t)^2 - \alpha (\alpha - 1) x^{\alpha-2} (x^\alpha - t^\alpha) (x - t)^2.$$  \hfill (27)

Writing $s = tx$, we see that

$$\Delta(x, t) = x^{2\alpha} H(s),$$

where

$$H(s) := (1 - s^\alpha)^2 + \alpha^2 (1 - s)^2 - \alpha (\alpha - 1) (1 - s^\alpha) (1 - s)^2.$$  \hfill (27)

For $s > 1$, all three terms in the right-hand side of (27) are positive, so $H(s) > 0$. If $0 \leq s < 1$, we see that

$$H(s) = (1 - s^\alpha)^2 + \alpha (1 - s)^2 \{\alpha - (\alpha - 1) (1 - s^\alpha)\}$$

$$\geq (1 - s^\alpha)^2 + \alpha (1 - s)^2 > 0.$$
In summary, if $\alpha > 1$, we have for all $x \in [0, 1]$ and $s \in [0, \infty) \setminus \{1\}$,

$$\Delta (x, sx) > 0$$

so the hypotheses (24) is fulfilled.

**Case II** $\alpha < 1$

Here

$$\psi^{-1} (x) = x^{1/\alpha}$$

and

$$K (\psi^{-1} (x), \psi^{-1} (t)) = \log \frac{1}{|x^{1/\alpha} - t^{1/\alpha}|} + \log \frac{1}{|x - t|},$$

which is exactly the case $1/\alpha > 1$ treated above, so we see that the hypothesis (25) is fulfilled.

Instead of placing an implicit assumption on the support of $\nu_\psi$, we can place an implicit assumption on the zeros of $\{P_n\}$, and obtain a unique weak limit:

**Theorem 1.5.** Let $\psi : [0, 1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function that preserves smallness of sets. Let $K \subset [0, 1]$ be compact. Assume that every weak limit of every subsequence of the zero counting measures $\{\mu_n\}$ has support $K$. Then there is a unique probability measure $\mu$ on $K$ such that

$$\mu_n \xrightarrow{s} \mu, \ n \to \infty,$$

and a unique positive number $A$ such that

$$\lim_{n \to \infty} n^{1/n} = A.$$  

Here $\mu$ is absolutely continuous with respect to linear Lebesgue measure, and is the unique solution of the integral equation

$$W^\mu (x) = \text{Constant}, \quad \text{q.e.} \ x \in K,$$

Moreover, then

$$W^\mu (x) = \log \frac{1}{A}, \quad \text{q.e.} \ x \in K.$$

We note that in [6], a related integral equation to (30) appears. We shall also need the dual polynomials $Q_n$ such that $Q_n \circ \psi$ are biorthogonal
to powers of $x$. Thus we define $Q_n$ to be a monic polynomial of degree $n$ determined by the conditions

$$\int_0^1 Q_n \circ \psi (t) t^j dt = 0, \quad j = 0, 1, 2, \ldots, n - 1.$$  (31)

Because of this biorthogonality condition,

$$\int_0^1 Q_n \circ \psi (t) t^n dt = \int_0^1 Q_n \circ \psi (t) P_n (t) dt = \int_0^1 P_n (t) \psi (t)^n dt.$$  (32)

That is,

$$I_n = \int_0^1 P_n (t) \psi (t)^n dt = \int_0^1 Q_n \circ \psi (t) t^n dt.$$  (32)

The orthogonality conditions ensure that $Q_n \circ \psi$ has $n$ distinct zeros $\{y_{jn}\}$ in $(0, 1)$, so we can write

$$Q_n \circ \psi (t) = \prod_{j=1}^n (\psi (t) - \psi (y_{jn})).$$  (33)

Let

$$\nu_n = \frac{1}{n} \sum_{j=1}^n \delta_{y_{jn}}.$$  (34)

We shall prove

**Theorem 1.6.** Let $\psi : [0, 1] \to [\psi (0), \psi (1)]$ be a strictly increasing continuous function that preserves smallness of sets, and assume (21). We have as $n \to \infty$,

$$\nu_n \overset{\ast}{\rightarrow} \nu_\psi.$$  

We also prove the following extremal property for weak subsequential limits of $\{\mu_n\}$.

**Theorem 1.7.** Let $\psi : [0, 1] \to [\psi (0), \psi (1)]$ be a strictly increasing continuous function that preserves smallness of sets. Assume that $S$ is an infinite subsequence of positive integers such that as $n \to \infty$ through $S$,

$$\mu_n \overset{\ast}{\rightarrow} \mu; \quad \nu_n \overset{\ast}{\rightarrow} \nu;$$  (35, 36)

and

$$I_n^{1/n} \to A.$$  (37)
where $A \in \mathbb{R}$ and $\mu, \nu \in \mathcal{P}([0,1])$. Then

$$A \leq \exp \left( - \sup_{\beta \in \mathcal{P}([0,1])} \inf_{[0,1]} W^{\mu,\beta} \right)$$

and

$$A \leq \exp \left( - \sup_{\alpha \in \mathcal{P}([0,1])} \inf_{[0,1]} W^{\alpha,\nu} \right).$$

**Remarks.**

(a) This extremal property is very close to a characterization of equilibrium measures for external fields. For example, with $\nu$ as above, let $Q$ be the external field

$$Q = U^{\nu \circ \psi^{-1}} \circ \psi \text{ on } [0,1].$$

Then the second inequality above says

$$A \leq \exp \left( - \sup_{\alpha \in \mathcal{P}([0,1])} \inf_{[0,1]} (U^{\alpha} + Q) \right).$$

This is reminiscent of one characterization of the equilibrium measure for the external field $Q$ [20, Theorem I.3.1, p. 43].

(b) Herbert Stahl sketched a proof that when $\psi$ is strictly increasing and piecewise linear, then (21) holds [27]. His expectation was that this and a limiting argument could establish (21) very generally.

(c) There are two principal issues left unresolved in this paper, that seem worthy of further study:

(I) Find general hypotheses for $\text{supp}[\nu \circ \psi] = [0,1]$.

(II) Find an explicit representation of the solution $\mu'$ of the integral equation (30), that is of

$$\int_0^1 \log |x - t| \mu'(t) \, dt$$

$$+ \int_0^1 \log |\psi(x) - \psi(t)| \mu'(t) \, dt = \text{Constant}, \quad x \in [0,1].$$

The usual methods (differentiating, and solving a Cauchy singular integral equation) do not seem to work, even when $\psi$ is analytic.
Next we show that if $\psi$ is constant in an interval, then the support of the equilibrium measure should avoid that interval, as do most of the zeros of $\{P_n\}$:

**Example.** Let

$$
\psi(x) = \begin{cases} 
2x, & x \in [0, \frac{1}{2}] \\
1, & x \in [\frac{1}{2}, 1] 
\end{cases}.
$$

Then it is not difficult to see that the equilibrium measure $\nu_\psi$ must have support $[0, \frac{1}{2}]$. Indeed if $\mu$ is a probability measure that has positive measure on $[a,b] \subset (\frac{1}{2}, 1)$, then as

$$
\log \frac{1}{|\psi(x) - \psi(t)|} = \infty, \quad x, t \in [a,b],
$$

so

$$
J(\mu) = \infty.
$$

Consequently,

$$
J^* = \inf \left[ 2I(\mu) + \log \frac{1}{2} \right],
$$

where the inf is now taken over all $\mu \in \mathcal{P}([0, \frac{1}{2}])$. Then $\nu_\psi$ is the classical equilibrium measure for $[0, \frac{1}{2}]$, namely

$$
\nu_\psi'(x) = \frac{1}{\pi \sqrt{x \left( \frac{1}{2} - x \right)}}, \quad x \in \left[ 0, \frac{1}{2} \right],
$$

and

$$
J^* = 2 \log 8 + \log \frac{1}{2} = \log 32.
$$

In this case, we can also almost explicitly determine $P_n$. The biorthogonality conditions give for $\pi$ of degree at most $n - 1$,

$$
\int_0^{1/2} P_n(x) \pi(2x) dx + \pi(1) \int_{1/2}^1 P_n(x) dx = 0.
$$

In particular, this is true for $\pi \equiv 1$, so

$$
\int_{1/2}^1 P_n(x) dx = - \int_0^{1/2} P_n(x) dx,
$$

and we obtain for any $\pi$ of degree at most $n - 1$,

$$
\int_0^{1/2} P_n(x) (\pi(2x) - \pi(1)) dx = 0.
$$
Then for every polynomial $S$ of degree $\leq n - 2$,
\[
\int_0^{1/2} P_n(x) S(x) (1 - 2x) \, dx = 0,
\] (40)
which forces $P_n$ to have at least $n - 1$ distinct zeros in $[0, \frac{1}{2}]$. Then every weak limit of every subsequence of $\{\mu_n\}$ has support in $[0, \frac{1}{2}]$.

This paper is organized as follows: in Section 2, we present a principle of descent, and a lower envelope theorem, and the proof of Theorem 1.2. In Section 3, we prove Theorems 1.3–1.7. Throughout the sequel, we assume that $\psi : [0, 1] \to [\psi(0), \psi(1)]$ is a strictly increasing continuous function that preserves smallness of sets.

We close this section with some extra notation. Define the **companion polynomial** to $P_n$, namely
\[
R_n(x) = \prod_{j=1}^{n} (x - \psi(x_j)).
\] (41)
It has the property that $R_n \circ \psi$ has the same zeros as $P_n$. Hence
\[
P_n(x) R_n \circ \psi(x) \geq 0 \text{ in } [0, 1].
\] (42)

Analogous to $R_n$, we define
\[
S_n(t) = \prod_{j=1}^{n} (t - y_j),
\] (43)
so that
\[
S_n(t) Q_n \circ \psi(t) \geq 0, \quad t \in [0, 1].
\] (44)
Observe that $I_n$ of (20) satisfies
\[
I_n = \int_0^1 P_n(x) R_n \circ \psi(x) \, dx = \int_0^1 Q_n \circ \psi(x) S_n(x) \, dx > 0.
\] (45)

2. **Proof of Theorem 1.2**

We begin by noting that for any positive measures $\alpha, \beta$, $W^{\alpha, \beta}$ is lower semicontinuous, since a potential of any positive measure is, while $\psi$ and $\psi^{-1}$ are continuous. We start with
Lemma 2.1 (The Principle of Descent). Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be finite positive Borel measures on \([0, 1]\) such that
\[
\lim_{n \to \infty} \alpha_n ([0, 1]) = 1 = \lim_{n \to \infty} \beta_n ([0, 1]).
\]
Assume moreover that as \( n \to \infty \),
\[
\alpha_n \rightharpoonup \alpha; \\
\beta_n \rightharpoonup \beta.
\]
(a) If \( \{x_n\} \subset [0, 1] \) and \( x_n \to x_0, n \to \infty \), then
\[
\liminf_{n \to \infty} W_{\alpha_n, \beta_n} (x_n) \geq W_{\alpha, \beta} (x_0).
\]
(b) If \( K \subset [0, 1] \) is compact and
\[
W_{\alpha, \beta} \geq \lambda \text{ in } K,
\]
then uniformly in \( K \),
\[
\liminf_{n \to \infty} W_{\alpha_n, \beta_n} (x) \geq \lambda.
\]

Proof. (a) By the classical principle of descent,
\[
\liminf_{n \to \infty} U_{\alpha_n} (x_n) \geq U_{\alpha} (x_0),
\]
see for example, [20, Theorem I.6.8, p. 70]. Next, we see from the classical principle of descent and continuity of \( \psi, \psi^{-1} \) that
\[
\liminf_{n \to \infty} U_{\beta_n o \psi^{-1}} \circ \psi (x_n) \geq U_{\beta o \psi^{-1}} \circ \psi (x_0).
\]
Combining these two gives the result.

(b) This follows easily from (a). If (b) fails, we can choose a sequence \( \{x_n\} \) in \( K \) with limit \( x_0 \in K \) such that
\[
\liminf_{n \to \infty} W_{\alpha_n, \beta_n} (x_n) < \lambda \leq W_{\alpha, \beta} (x_0).
\]

Recall our notation \( W^{\alpha_n} = W_{\alpha_n, \alpha_n} \). We now establish
Lemma 2.2 (Lower Envelope Theorem). Assume the hypotheses of Lemma 2.1. Then for q.e. \( x \in [0, 1] \),

\[
\liminf_{n \to \infty, n \in S} W^{\alpha_n}(x) = W^\alpha(x).
\]

Proof. We already know from Lemma 2.1 (the principle of descent) that everywhere in \([0, 1]\),

\[
\liminf_{n \to \infty, n \in S} W^{\alpha_n}(x) \geq W^\alpha(x).
\]

Suppose the result is false. Then there exists \( \varepsilon > 0 \), and a (Borel) set \( S \) of positive capacity such that

\[
\liminf_{n \to \infty, n \in S} W^{\alpha_n}(x) \geq W^\alpha(x) + \varepsilon \quad \text{in } S.
\]

Because Borel sets are inner regular, and even more, capacitable, we may assume that \( S \) is compact. Then there exists a probability measure \( \omega \) with support in \( S \) such that \( U^\omega \) is continuous in \( \mathbb{C} \). See, for example, [20, Corollary I.6.11, p. 74]. As \( \psi \) and \( \psi_{[-1]} \) are continuous,

\[
W^\omega = U^\omega + U^\omega \psi_{[-1]} \circ \psi
\]

is also continuous in \([0, 1]\). Then by Fubini’s Theorem and weak convergence

\[
\liminf_{n \to \infty, n \in S} \int W^{\alpha_n} d\omega = \liminf_{n \to \infty, n \in S} \int W^\omega d\alpha_n
\]

\[
= \int W^\omega d\alpha = \int W^\alpha d\omega.
\]

Here since \( K(x, t) \) is bounded below in \([0, 1]\), we may continue this using (46) and Fatou’s Lemma as

\[
= \int (W^\alpha + \varepsilon) d\omega - \varepsilon
\]

\[
\leq \int \left( \liminf_{n \to \infty, n \in S} W^{\alpha_n} \right) d\omega - \varepsilon
\]

\[
\leq \liminf_{n \to \infty, n \in S} \int W^{\alpha_n} d\omega - \varepsilon.
\]

So we have a contradiction. \( \Box \)

Next, we show that \( J^* \) is finite, establishing part of Theorem 1.2(a):
Lemma 2.3. $J^*$ is finite.

Proof. This is really a consequence of Cartan’s Lemma for potentials. Let $\mu = \text{meas}$ denote Lebesgue measure on $[0, 1]$. Then for $x \in [0, 1]$, 

$$U^\mu (x) = \int_0^1 \log \frac{1}{|x-t|} dt \leq 2 \int_0^1 \log \frac{1}{s} ds$$

and $U^\mu$ is continuous. Now consider the unit measure $\mu \circ \psi^{-1}$. By Cartan’s Lemma [9, p. 366], if $\varepsilon > 0$ and 

$$\mathcal{A}^\varepsilon = \left\{ y \in \mathbb{R} : U^{\mu \circ \psi^{-1}} (y) > \log \frac{1}{\varepsilon} \right\},$$

then 

$$\mu (\mathcal{A}^\varepsilon) \leq 3 \varepsilon \varepsilon.$$

With a suitably small choice of $\varepsilon$, we then have by the hypothesis (13), 

$$\mu \left( \psi^{-1} (\mathcal{A}^\varepsilon) \right) \leq \frac{1}{2}.$$

With this choice of $\varepsilon$, let 

$$\mathcal{B} = [0, 1] \setminus \psi^{-1} (\mathcal{A}^\varepsilon),$$

a closed set. Let 

$$\nu = \frac{\mu |_{\mathcal{B}}}{\mu (\mathcal{B})}.$$

As $\mu (\mathcal{B}) \geq \frac{1}{2}$, $\nu$ is a well defined probability measure. Moreover, $x \in \mathcal{B} \Rightarrow \psi (x) \notin \mathcal{A}^\varepsilon$, and 

$$U^{\nu \circ \psi^{-1}} \circ \psi (x) = \frac{1}{\mu (\mathcal{B})} \left[ U^{\mu \circ \psi^{-1}} \circ \psi (x) - U^{\mu |_{[0,1] \setminus \psi^{-1}} \circ \psi (x)} \right]$$

$$\leq \frac{1}{\mu (\mathcal{B})} \left[ \log \frac{1}{\varepsilon} + \log \left( 2 \| \psi \|_{L_\infty [0,1]} \right) \right] =: C_0 < \infty.$$ 

Then 

$$J^* \leq J (\nu) \leq I (\nu) + C_0 < \infty.$$

□
Proof of Theorem 1.2. (a) We can choose a sequence \( \{ \alpha_n \} \) of probability measures on \([0, 1]\) such that
\[
\lim_{n \to \infty} J(\alpha_n) = J^*.
\]
By Helly’s Theorem, we can choose a subsequence converging weakly to some probability measure \( \alpha \) on \([0, 1]\), and by relabelling, we may assume that the full sequence \( \{ \alpha_n \} \) converges weakly to \( \alpha \). Then \( \{ \alpha_n \circ \psi[-1] \} \) converges weakly to \( \alpha \circ \psi[-1] \). By the classical principle of descent
\[
\lim_{n \to \infty} I(\alpha_n) \geq I(\alpha)
\]
and
\[
\lim_{n \to \infty} I(\alpha_n \circ \psi[-1]) \geq I(\alpha \circ \psi[-1]),
\]
or equivalently,
\[
\lim_{n \to \infty} \int \int \frac{1}{|\psi(x) - \psi(t)|} d\alpha_n(x) d\alpha_n(t) \geq \int \int \frac{1}{|\psi(x) - \psi(t)|} d\alpha(x) d\alpha(t).
\]
See, for example, [20, Thm. I.6.8, p. 70]. Combining these, we have
\[
J^* = \lim_{n \to \infty} J(\alpha_n) \geq J(\alpha),
\]
so \( \alpha \) achieves the inf, and is an equilibrium distribution. If \( \beta \) is another such distribution, then the parallelogram law
\[
J\left(\frac{1}{2} (\alpha + \beta)\right) + J\left(\frac{1}{2} (\alpha - \beta)\right) = \frac{1}{2} (I(\alpha) + J(\beta)) = J^*,
\]
gives
\[
J\left(\frac{1}{2} (\alpha - \beta)\right) = J^* - J\left(\frac{1}{2} (\alpha + \beta)\right) \leq 0,
\]
as \( \frac{1}{2} (\alpha + \beta) \) is also a probability measure on \([0, 1]\). Here
\[
J\left(\frac{1}{2} (\alpha - \beta)\right) = I\left(\frac{1}{2} (\alpha - \beta)\right) + I\left(\frac{1}{2} (\alpha \circ \psi[-1] - \beta \circ \psi[-1])\right),
\]
and both terms on the right-hand side are non-negative as both measures inside the energy integrals on the right have total mass 0. See [20, Lemma I.1.8, p. 29]. Hence
\[
I\left(\frac{1}{2} (\alpha - \beta)\right) = 0,
\]
so $\alpha = \beta$ [20, Lemma I.1.8, p. 29].

(b) Suppose the result is false. Then for some large enough integer $n_0$,

$$E_1 := \left\{ x \in [0, 1] : W^{\nu_\psi} (x) \leq J^* - \frac{1}{n_0} \right\},$$

has positive capacity and is compact, since $W^{\nu_\psi}$ is lower semi-continuous. But,

$$\int W^{\nu_\psi} d\nu_\psi = J (\nu_\psi) = J^*, \quad so there exists a compact subset $E_2$ disjoint from $E_1$ such that

$$W^{\nu_\psi} (x) > J^* - \frac{1}{2n_0}, \quad x \in E_2,$$

and

$$m = \nu_\psi (E_2) > 0.$$

Now as $E_1$ is a compact set of positive capacity, we can find a positive measure $\sigma$ on $E_1$, with support in $E_1$, such that $U^\sigma$ is continuous in the plane [20, Cor. I.6.11, p. 74]. Then $U^{\sigma \circ \psi_{[-1]}}$ is also continuous in $[\psi (0), \psi (1)]$, so $W^\sigma$ is continuous in $[0, 1]$. We may also assume that

$$\sigma (E_1) = m.$$

Define a signed measure $\sigma_1$ on $[0, 1]$, by

$$\sigma_1 := \begin{cases} \sigma & \text{in } E_1 \\ -\nu_\psi & \text{in } E_2 \\ 0 & \text{elsewhere} \end{cases}.$$

Here if $\eta \in (0, 1)$,

$$J (\nu_\psi + \eta \sigma_1) = J (\nu_\psi) + 2\eta \int W^{\nu_\psi} d\sigma_1 + \eta^2 J (\sigma_1)$$

$$\leq J (\nu_\psi) + 2\eta \left\{ \int_{E_1} \left[ J^* - \frac{1}{n_0} \right] d\sigma + \int_{E_2} \left[ J^* - \frac{1}{2n_0} \right] d(-\nu_\psi) \right\} + \eta^2 J (\sigma_1)$$

$$= J (\nu_\psi) + 2\eta m \left\{ \left[ J^* - \frac{1}{n_0} \right] - \left[ J^* - \frac{1}{2n_0} \right] \right\} + \eta^2 J (\sigma_1)$$

$$= J (\nu_\psi) - \frac{\eta m}{n_0} + \eta^2 J (\sigma_1) < J (\nu_\psi),$$

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for small $\eta > 0$. As $\sigma_1$ has total mass 0, so $\nu_\psi + \eta \sigma_1$ has total mass 1, and we see from the identity
\[
\nu_\psi + \eta \sigma_1 = (1 - \eta) \nu_\psi|_{E_2} + \nu_\psi|_{[0,1]\setminus E_2} + \eta \sigma
\]
that it is non-negative. Then we have a contradiction to the minimality of $J(\nu_\psi)$.

(c) Let $x_0 \in \text{supp}[\nu_\psi]$ and suppose that
\[
W^{\nu_\psi}(x_0) > J^*.
\]
By lower semi-continuity of $W^{\nu_\psi}$, there exists $\epsilon > 0$ and closed $[a, b]$ containing $x_0$ such that
\[
W^{\nu_\psi}(x) > J^* + \epsilon, \quad x \in [a, b].
\]
We know too that
\[
W^{\nu_\psi}(x) \geq J^* \text{ for q.e. } x \in \text{supp}[\nu_\psi].
\]
Here as $J^*$ is finite, so $I(\nu_\psi)$ must be finite (recall that $K(x,t)$ is bounded below). Then $\nu_\psi$ vanishes on sets of capacity 0, so this last inequality holds $\nu_\psi$ a.e. (cf. [19, Theorem 3.2.3, p. 56]). Then
\[
J^* = J(\nu_\psi) = \left( \int_a^b + \int_{[0,1]\setminus[a,b]} \right) W^{\nu_\psi}(x) \, d\nu_\psi(x)
\]
\[
\geq (J^* + \epsilon) \nu_\psi([a,b]) + J^* \nu_\psi([0,1]\setminus[a,b])
\]
\[
= J^* + \epsilon \nu_\psi([a,b]),
\]
a contradiction.

(d) If $\text{cap}(K) = 0$, then as $I(\nu_\psi) < \infty$, we have also $\nu_\psi(K) = 0$, and the inequality (18) is immediate. So assume that $K \subseteq \text{supp}[\nu_\psi]$ has positive capacity, and let $\omega$ be the equilibrium measure for $K$. We may also assume that $K \subseteq \text{supp}[\nu_\psi]$, since
\[
\nu_\psi(K) = \nu_\psi(K \cap \text{supp}[\nu_\psi]).
\]
Now, there exists a positive constant $C_0$ such that
\[
K(x,t) \geq -C_0, \quad x, t \in [0,1].
\]
Then by (c), for $x \in \mathcal{K}$,
\[
\int_{\mathcal{K}} K(x,t) \, d\nu_\psi(t) \leq J^* - \int_{[0,1] \setminus \mathcal{K}} K(x,t) \, d\nu_\psi(t) \\
\leq J^* + C_0
\]
and hence for $x \in \mathcal{K}$,
\[
\int_{\mathcal{K}} \log \frac{1}{|x-t|} \, d\nu_\psi(t) \leq J^* + C_0 + \log \left(2\|\psi\|_{L^\infty([0,1])}\right) =: C_1.
\] (47)
Here $C_1$ is independent of $\mathcal{K}, x$. Now
\[
U^\omega(t) = \log \frac{1}{\text{cap} \mathcal{K}}
\]
for q.e. $t \in \mathcal{K}$ and since $\nu_\psi$ vanishes on sets of capacity zero, this also holds for $\nu_\psi$ a.e. $t \in \mathcal{K}$. Integrating (47) with respect to $d\omega(x)$ and using Fubini’s theorem, gives
\[
\int_{\mathcal{K}} U^\omega(t) \, d\nu_\psi(t) \leq C_1
\]
and hence
\[
\nu_\psi(\mathcal{K}) \log \frac{1}{\text{cap} \mathcal{K}} \leq C_1.
\]
This gives the first inequality in (18), and then well known inequalities relating $\text{cap}$ and $\text{meas}$ give the second. In particular, that inequality implies the absolute continuity of $\mu$ with respect to linear Lebesgue measure.

(e) Suppose that $0 \notin \text{supp} [\nu_\psi]$. Let $c > 0$ be the closest point in the support of $\nu_\psi$ to 0. Then for $x \in [0, \frac{c}{2}]$, and for all $t \in [c, 1]$, we have from the strict monotonicity of $\psi$ that
\[
K(x,t) < K(c,t),
\]
so for such $x$,
\[
W^{\nu_\psi}(x) = \int_c^1 K(x,t) \, d\nu_\psi(t) \\
< \int_c^1 K(c,t) \, d\nu_\psi(t) = W^{\nu_\psi}(c) \leq J^*.
\]
Thus in spite of the continuity of $W^{\nu_\psi}$ in $[0,c)$,
\[
W^{\nu_\psi} < J^* \text{ in } \left[0, \frac{c}{2}\right],
\]
contradicting (b). Absolute continuity of $\nu_\psi$ then shows that for some $\varepsilon > 0$, we have $[0,\varepsilon] \subset \text{supp} [\nu_\psi]$. Similarly we can show that for some $\varepsilon > 0$, $[1-\varepsilon,1] \subset \text{supp} [\nu_\psi]$. \qed
3. Proof of Theorems 1.3–1.7

Recall that $\mu_n$ and $\nu_n$ were defined respectively by (3) and (34). Throughout this section, we assume that $S$ is an infinite subsequence of positive integers such that as $n \to \infty$ through $S$,

$$
\mu_n \overset{*}{\to} \mu; \\
\nu_n \overset{*}{\to} \nu;
$$

and

$$I_n^{1/n} \to A,
$$

where $A \in \mathbb{R}$ and $\mu, \nu \in \mathcal{P}([0,1])$. In the sequel we make frequent use of identities such as

$$|P_n(x)|^{1/n} = \exp(-U_{\mu_n}(x))
$$

and

$$|P_n(x) R_n \circ \psi(x)|^{1/n} = \exp(-W_{\mu_n}(x)).
$$

We begin with

**Lemma 3.1** (An upper bound for $W^\mu$). *(a) With the hypotheses above, let $[a,b] \subset [0,1]$ and assume that $[a,b]$ contains two zeros of $P_n$ for infinitely many $n \in S$. Then

$$\inf_{[a,b]} W^\mu \leq \log \frac{1}{A}.
$$

(b) In particular, if $x_0$ is a limit of two zeros of $P_n$ as $n \to \infty$ through $S$, or $x_0 \in \text{supp } [\mu]$, then

$$W^\mu(x_0) \leq \log \frac{1}{A}.
$$

**Proof.** *(a) We may assume (by passing to a subsequence) that for all $n \in S$, $P_n$ has two zeros in $[a,b]$. Assume on the contrary, that for some $\varepsilon > 0$,

$$\inf_{[a,b]} W^\mu > \log \frac{1}{A} + \varepsilon. \tag{51}
$$

Let $x_n, y_n$ be two zeros of $P_n$ in $[a,b]$ and let

$$R_n^*(x) = R_n(x) / [(x - \psi(x_n))(x - \psi(y_n))].
$$

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Then we see that
\[ P_n(x) R_n^* \circ \psi(x) \geq 0, \quad x \in [0,1] \setminus [a,b], \]
and
\[ 0 \leq P_n(x) R_n \circ \psi(x) \leq |P_n(x) R_n^* \circ \psi(x)| \left( 4\|\psi\|_{L_\infty[0,1]} \right)^2, \quad x \in [0,1]. \]
Moreover, as \( R_n^* \) has the same asymptotic zero distribution as \( R_n \), we see from Lemma 2.1 and (51) that
\[
\limsup_{n \to \infty, n \in S} \left( \int_{0,1} |P_n(x) R_n^* \circ \psi(x)| \, dx \right)^{1/n} \leq \exp \left( -W^{\mu,\mu}(x) \right) = \exp \left( -W^{\mu}(x) \right) \leq Ae^{-\varepsilon},
\]
uniformly in \([a,b]\). Then by biorthogonality, and positivity of \( P_n(x) R_n^* \circ \psi(x) \) outside \([a,b]\),
\[
\limsup_{n \to \infty, n \in S} \left( \int_{[0,1] \setminus [a,b]} |P_n(x) R_n^* \circ \psi(x)| \, dx \right)^{1/n} \leq Ae^{-\varepsilon}.
\]
Of course Lemma 2.1(b) also gives
\[
\limsup_{n \to \infty, n \in S} \left( \int_{[a,b]} |P_n(x) R_n^* \circ \psi(x)| \, dx \right)^{1/n} \leq Ae^{-\varepsilon},
\]
so
\[
A = \limsup_{n \to \infty, n \in S} I_n^{1/n} \\
\leq \limsup_{n \to \infty, n \in S} \left( 4\|\psi\|_{L_\infty[0,1]} \right)^{2/n} \left( \int_{0}^{1} |P_n(x) R_n^* \circ \psi(x)| \, dx \right)^{1/n} \\
\leq Ae^{-\varepsilon}.
\]
This contradiction gives the result.
(b) This follows from (a), and lower semicontinuity of \( W^{\mu}. \)

**Lemma 3.2** (A Lower bound for \( W^{\mu}. \)). *At each point of continuity of \( W^{\mu} \) in \([0,1]\), we have*

\[ W^{\mu} \geq \log \frac{1}{A}. \]  \tag{52}

*In particular, this inequality holds q.e. in \([0,1]\).*
Proof. Assume that $a \in [0, 1]$ is a point of continuity of $W^\mu$, but for some $\varepsilon > 0$,

$$W^\mu(a) \leq \log \frac{1}{A} - 2\varepsilon.$$ 

Then there exists an interval $[a, b]$ containing $a$, such that

$$W^\mu(x) \leq \log \frac{1}{A} - \varepsilon, \quad x \in [a, b].$$

By the lower envelope theorem (Lemma 2.2)

$$\limsup_{n \to \infty, n \in S} \left( P_n(x) R_n \circ \psi(x) \right)^{1/n}$$

$$= \exp \left( - \liminf_{n \to \infty, n \in S} W^{\mu_n}(x) \right) = \exp (-W^\mu(x)) \geq Ae^\varepsilon$$

for q.e. $x \in [a, b]$. Let

$$T_n = \left\{ x \in [a, b] : \left( P_n(x) R_n \circ \psi(x) \right)^{1/n} \geq Ae^{\varepsilon/2} \right\}.$$ 

Then for each $m \geq 1$,

$$\bigcup_{n=m}^{\infty} T_n$$

contains q.e. $x \in [a, b]$, so has linear Lebesgue measure $b - a$. Then for infinitely many $n$, $T_n$ has linear Lebesgue measure at least $n^{-2}$, so

$$I_n^{1/n} \geq \left( \int_{T_n} P_n(x) R_n \circ \psi(x) \, dx \right)^{1/n}$$

$$\geq n^{-2/n} Ae^{\varepsilon/2}$$

so

$$A = \limsup_{n \to \infty, n \in S} I_n^{1/n} \geq Ae^{\varepsilon/2},$$

a contradiction.

Finally, we note that any logarithmic potential is continuous q.e. [13, p. 185], so $U^\mu$ and $U^{\mu \circ \psi[-1]}$ are continuous q.e. Our hypothesis that $\psi[-1](E)$ has capacity zero whenever $E$ does ensures that $U^{\mu \circ \psi[-1]} \circ \psi$ is continuous q.e. also. Hence $W^\mu$ is continuous q.e. and so (52) holds q.e. in $[0, 1]$. 

Next, we establish lower and upper bounds for $A$. 21
Lemma 3.3. (a) There exist constants $C_1, C_2 > 0$ depending only on $\psi$ (and not on the subsequence $S$ above) such that

$$C_1 \geq A \geq C_2. \quad (53)$$

(b) In particular,

$$I(\mu) < \infty.$$  

(c) 

$$J(\mu) = \log \frac{1}{A}$$  

and

$$W^\mu = \log \frac{1}{A} \text{ q.e. and a.e.} \ (\mu) \text{ in supp}[\mu]. \quad (55)$$

(d) $\mu$ is absolutely continuous with respect to linear Lebesgue measure on $[0,1]$. Moreover, there are constants $C_1$ and $C_2$ depending only on $\psi$, and not on $S$, such that for all compact $K \subset [0,1]$,

$$\mu(K) \leq \frac{C_1}{|\log \text{cap} K|} \leq \frac{C_2}{|\log \text{meas} (K)|}.$$ 

Proof. (a) Firstly as all zeros of $P_n$ and $R_n \circ \psi$ lie in $[0,1]$, so

$$I_n = \int_0^1 P_n(x) R_n \circ \psi (x) \, dx$$

$$\leq (\text{diam } [0,1])^n.$$ 

Here diam denotes the diameter of a set. So

$$A \leq \text{diam } [0,1].$$

In the other direction, we use Cartan’s Lemma for polynomials [2, p. 175], [9, p. 366]. This asserts that if $\delta > 0$, then

$$|R_n (x)| \geq \left( \frac{\delta}{4e} \right)^n$$

outside a set $\mathcal{E}$ of linear Lebesgue measure at most $\delta$. Then

$$|R_n \circ \psi (x)| \geq \left( \frac{\delta}{4e} \right)^n, \quad x \in [0,1] \setminus \psi([-1]) (\mathcal{E}).$$

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By our hypothesis (13), we may choose \( \delta \) so small that
\[
\text{meas} (\mathcal{E}) \leq \delta \Rightarrow \text{meas} \left( \psi^{-1} (\mathcal{E}) \right) \leq \frac{1}{4}.
\]
Next, Cartan’s Lemma also shows that
\[
|P_n (x)| \geq \left( \frac{1}{16e} \right)^n, \quad x \in [0, 1] \setminus \mathcal{F},
\]
where
\[
\text{meas} (\mathcal{F}) \leq \frac{1}{4}.
\]
Then
\[
P_n (x) R_n \circ \psi (x) \geq \left( \frac{\delta}{64e^2} \right)^n, \quad x \in [0, 1] \setminus \left( \psi^{-1} (\mathcal{E}) \cup \mathcal{F} \right)
\]
and so
\[
I_n \geq \int_{[0,1] \setminus (\psi^{-1} (\mathcal{E}) \cup \mathcal{F})} P_n (x) R_n \circ \psi (x) \, dx \geq \left( \frac{\delta}{64e^2} \right)^n \frac{1}{2}.
\]
Hence
\[
A \geq \frac{\delta}{64e^2}.
\]
(b) Since for \( x, t \in [0, 1] \),
\[
\log \frac{1}{|\psi (x) - \psi (t)|} \geq \log \frac{1}{2 \text{diam} \psi [0, 1]} > -\infty,
\]
so for \( x \in \text{supp} [\mu] \), Lemma 3.1(b) gives
\[
\log \frac{1}{A} \geq W^\mu (x) \geq \log \left[ \frac{1}{U^\mu (x)} + \frac{1}{2 \text{diam} \psi [0, 1]} \right].
\]
Then
\[
I (u) \leq \log \frac{1}{A} - \log \frac{1}{2 \text{diam} \psi [0, 1]}.
\]
(c) As \( \mu \) has finite energy, it vanishes on sets of capacity zero. Then combining Lemma 3.1 and 3.2,
\[
W^\mu = \log \frac{1}{A} \text{ both q.e. and a.e. (} \mu \text{) in } \text{supp} [\mu].
\]
Then the first assertion (54) also follows.
(d) This is almost identical to that of Theorem 1.2(d), following from the fact that

\[ W^\mu \leq \log \frac{1}{A} \text{ in } \text{supp } [\mu]. \]

\[ \square \]

**Proof of Theorem 1.5.** Assume that \( S, \mu \) and \( A \) are as in the beginning of this section. Assume that \( S^#, \mu^#, A^# \) satisfy analogous hypotheses. We shall show that

\[ A = A^# \text{ and } \mu = \mu^#. \]

Our hypothesis on the zeros shows that

\[ \text{supp } [\mu] = \text{supp } [\mu^#] = \mathcal{K}. \]

Then Lemma 3.3 shows that

\[ W^\mu = \log \frac{1}{A} \text{ q.e. in } \mathcal{K} \]

and

\[ W^{\mu^#} = \log \frac{1}{A^#} \text{ q.e. in } \mathcal{K}. \]

Since \( I (\mu) \) and \( I (\mu^#) \) are finite by Lemma 3.3, these last statements also hold \( \mu \) a.e. and \( \mu^# \) a.e. in \( \mathcal{K} \). Then

\[ \log \frac{1}{A} = \int W^\mu d\mu^# = \int W^{\mu^#} d\mu = \log \frac{1}{A^#}. \]

It follows that there is a unique number \( A \) that is the limit of \( I_n^{1/n} \) as \( n \to \infty \). Next,

\[ J \left( \mu - \mu^# \right) = J (\mu) + J (\mu^#) - 2 \int W^\mu d\mu^# \]

\[ = \log \frac{1}{A} + \log \frac{1}{A} - 2 \log \frac{1}{A} = 0. \]

As in Theorem 1.2(a), this then gives

\[ \mu = \mu^#. \]

This proof also shows that \( \mu \) is the unique solution of the integral equation

\[ W^\mu = C \text{ q.e. in } \mathcal{K}. \]
We turn to the

**Proof of Theorem 1.3.** Let $\mu$ be a weak limit of some subsequence $\{\mu_n\}_{n \in S}$ of $\{\mu_n\}_{n=1}^{\infty}$. We may also assume that (50) holds. From Lemma 3.3, $\mu$ has finite logarithmic energy, and from Lemma 3.2,

$$W^\mu \geq \log \frac{1}{A} \text{ q.e. in } [0,1].$$

Moreover, by Theorem 1.2(c) and our hypothesis (21),

$$W^\nu \psi = J^* \text{ q.e. in } [0,1].$$

Then the last relations also hold $\mu$ a.e. and $\nu_\psi$ a.e., so

$$J^* = \int W^\nu \psi d\mu = \int W^\mu d\nu_\psi \geq \log \frac{1}{A}.$$

Moreover, by Lemma 3.3(c),

$$W^\mu = \log \frac{1}{A} \mu \text{ a.e. in } \text{supp } [\mu]$$

so

$$J(\mu) = \int W^\mu d\mu = \log \frac{1}{A} \leq J^*.$$ 

Then necessarily

$$\log \frac{1}{A} = J(\mu) = J^*$$

and

$$\mu = \nu_\psi.$$ 

**Proof of Theorem 1.4.** Assume first that $\psi''$ is continuous in $(0,1)$ and that for each $x, t \in [0,1]$ with $x \neq t$,

$$\frac{\partial^2}{\partial x^2} K(x, t) > 0,$$

but that the support is not all of $[0,1]$. We already know that $[0, \varepsilon] \cup [1 - \varepsilon, 1] \subset \text{supp } [\nu_\psi]$ for some $\varepsilon > 0$. Then there exist $0 < a < b < 1$ such that

$$(a, b) \cap \text{supp } [\nu_\psi] = \emptyset.$$  

(56)
We may assume that both

\[ a, b \in \text{supp } [\nu, \psi]. \]  

(57)

Then by Theorem 1.2(c),

\[ W^\nu \psi (a) \leq J^* \text{ and } W^\nu \psi (b) \leq J^*. \]

But in \( (a, b) \), which lies outside the support of \( \mu \), \( W^\mu \) will be twice continuously differentiable, and by our hypothesis,

\[ \frac{\partial^2}{\partial x^2} W^\nu \psi (x) = \int \frac{\partial^2}{\partial x^2} K (x, t) \, d\nu \psi (t) > 0. \]

The convexity of \( W^\nu \psi \) forces in some \( (c, d) \subset (a, b) \)

\[ W^\mu < J^*. \]

This contradicts Theorem 1.2(b).

Next, suppose that for \( x, t \in (\psi (0), \psi (1)) \) with \( x \neq t \),

\[ \frac{\partial^2}{\partial x^2} \left[ K \left( \psi^{-1} (x), \psi^{-1} (t) \right) \right] > 0. \]

Consider

\[ W^\nu \psi \circ \psi^{-1} (x) = \int K \left( \psi^{-1} (x), t \right) \, d\nu \psi (t) \]

\[ = \int K \left( \psi^{-1} (x), \psi^{-1} (s) \right) \, d\nu \psi \circ \psi^{-1} (s). \]

We have

\[ W^\nu \psi \circ \psi^{-1} (x) \leq J^* \quad \text{if } x \in \psi (\text{supp } [\nu, \psi]) \]

and at each point of continuity of \( W^\nu \psi \circ \psi^{-1} \), Theorem 1.2(b) gives

\[ W^\nu \psi \circ \psi^{-1} (x) \geq J^*. \]

We also see that for \( x \in [\psi (0), \psi (1)] \setminus \psi (\text{supp } [\nu, \psi]) \),

\[ \frac{\partial^2}{\partial x^2} \left[ W^\nu \psi \circ \psi^{-1} (x) \right] = \int \frac{\partial^2}{\partial x^2} \left[ K \left( \psi^{-1} (x), \psi^{-1} (s) \right) \right] \, d\nu \psi \circ \psi^{-1} (s) > 0. \]

If \( 0 < a < b < 1 \) and (56), (57) hold, then by Theorem 1.1(c),

\[ W^\nu \psi \circ \psi^{-1} (\psi (a)) \leq J^* \text{ and } W^\nu \psi \circ \psi^{-1} (\psi (b)) \leq J^* \]
so in some interval
\[ (c, d) \subset (\psi(a), \psi(b)), \]
the convexity gives
\[ W^{\nu} \circ \psi^{-1} < J^* \]
But then
\[ W^{\nu} < J^* \text{ in } (\psi(c), \psi(d)), \]
contradicting Theorem 1.2(b).
\[ \square \]

**Proof of Theorem 1.6.** Recall from (45) that
\[ I_n = \int_0^1 S_n Q_n \circ \psi \]
and
\[ |S_n(x) Q_n \circ \psi(x)|^{1/n} = \exp(-W^{\nu_\alpha}(x)). \]
Then much as in the proof of Lemma 3.1, 3.2, under the hypothesys (48)–(50), we obtain
\[ W^{\nu} \leq \log \frac{1}{A} \text{ in supp } [\nu] \]
and
\[ W^{\nu} \geq \log \frac{1}{A} \text{ q.e. in } [0, 1], \]
in particular at every point of continuity of \( W^{\nu} \). Then the proof of Theorem 1.3 shows that \( \nu = \nu_\psi \), and the result follows. \[ \square \]

We next prove an inequality for \( I_n \), assuming the hypotheses (35)–(36). Below, if \( \alpha, \beta \) are probability measures on \([0, 1]\), we set
\[ m_{\alpha, \beta} := \inf_{[0, 1]} W^{\alpha, \beta}. \]

**Proof of Theorem 1.7.** Let \( \beta \) be a probability measure on \([0, 1]\). By orthogonality, for any monic polynomial \( \Pi_n \) of degree \( n \), we have
\[ I_n = \int_0^1 P_n(x) \Pi_n \circ \psi(x) \, dx. \]
Given a probability measure on \([0, 1]\), we may choose a sequence of polynomials \( \Pi_n \) such that \( \Pi_n \) has \( n \) simple zeros in \([\psi(0), \psi(1)]\), and the corresponding zero counting measures converge weakly to \( \beta \circ \psi^{-1} \) as \( n \to \infty. \)
(This follows easily as pure jump measures are dense in the set of probability measures.) As

\[ W^{\mu,\beta} \geq m_{\mu,\beta} \]

in the closed set \([0, 1]\),
we obtain, by Lemma 2.1,

\[ \limsup_{n \to \infty, n \in S} \left| P_n(x) \Pi_n \circ \psi(x) \right|^{1/n} \leq \exp(-m_{\mu,\beta}), \]
uniformly in \([0, 1]\). Then

\[ A = \limsup_{n \to \infty, n \in S} I_n^{1/n} \leq \exp(-m_{\mu,\beta}). \]

Taking sup’s over all such \(\beta\) gives (38). The other relation follows similarly, because of the duality identity (32). \(\square\)

References


