The phase transition in bounded-size Achlioptas processes

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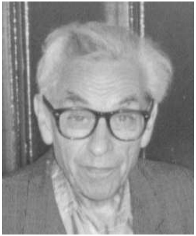
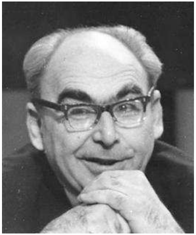
Joint work with Oliver Riordan



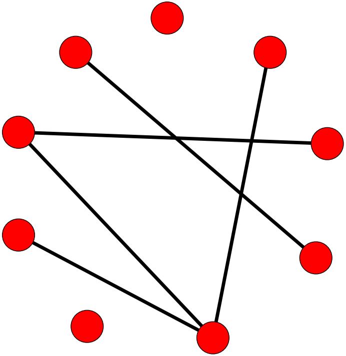
Paul Erdős



Alfred Rényi

Erdős–Rényi random graph process

- Start with an empty graph on n vertices
- In each step: add a random edge to the graph



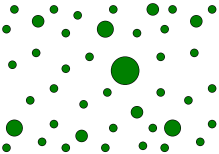
Erdős–Rényi random graph process

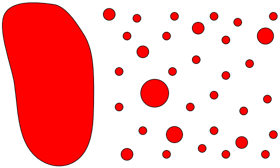
- Start with an empty graph on n vertices
- In each step: add a random edge to the graph

Phase transition (Erdős-Rényi, 1959)

Largest component 'dramatically changes' after $\approx n/2$ steps. Whp

$$L_1(tn) = \begin{cases} O(\log n) & \text{if } t < 1/2 \\ \Theta(n) & \text{if } t > 1/2 \end{cases}$$





Classical model

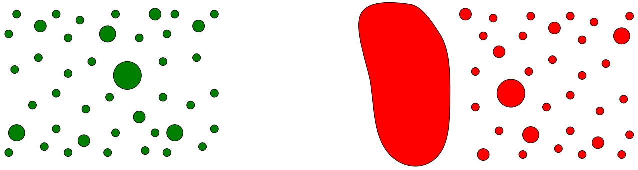
Erdős–Rényi random graph process

- Start with an empty graph on n vertices
- In each step: add a random edge to the graph

Phase transition (Erdős-Rényi, 1959)

Largest component 'dramatically changes' after $\approx n/2$ steps. Whp

$$L_1(tn) \approx \begin{cases} \varepsilon^{-2}\log(\varepsilon^3 n)/2 & \text{if } t = 1/2 - \varepsilon \\ 4\varepsilon n & \text{if } t = 1/2 + \varepsilon \end{cases}$$



Model with dependencies

Achlioptas processes

- Start with an empty graph on n vertices
- In each step: pick *two* random edges,
  add *one* of them to the graph (using some *rule*)

Remarks

- Yields *family* of random graph processes
- Contains 'classical' Erdős–Rényi process

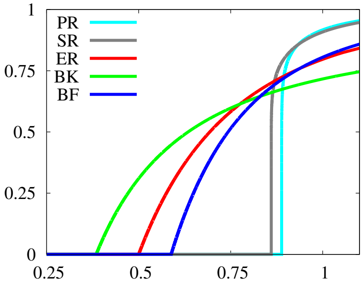
Motivation

- Improve our understanding of the phase transition phenomenon
- Test / develop methods for analyzing processes with dependencies

Phase transition in Achlioptas processes

Quantity of interest

Fraction of vertices in largest component after tn steps: $L_1(tn)/n$

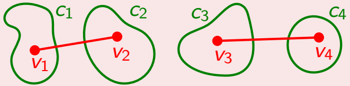


Goal of this talk

Prove that phase transition of a *large class* rules 'looks like' in Erdős–Rényi

Widely studied Achlioptas rules

Size rules



Decision (which edge to add) depends *only* on component *sizes* $c_1, \ldots, c_4$

- **Sum rule**: add $e_1 = \{v_1 v_2\}$ iff $c_1 + c_2 \leq c_3 + c_4$
  ('add the edge which results in the smaller component')

Bounded-size rules

All component sizes larger than some constant B are treated the same

- **Bohman–Frieze**: add $e_1 = \{v_1 v_2\}$ iff its endvertices are isolated
  ('add random edge with slight bias towards joining isolated vertices')

Previous work

Bounded-size rules (Spencer–Wormald, Bohman–Kravitz, Riordan–W.)

There is rule-dependent critical time $t_c > 0$ such that, whp,

$$L_1(tn) = \begin{cases} O(\log n) & \text{if } t < t_c \\ \Theta(n) & \text{if } t > t_c \end{cases}$$

Bohman–Frieze rule (Janson–Spencer)

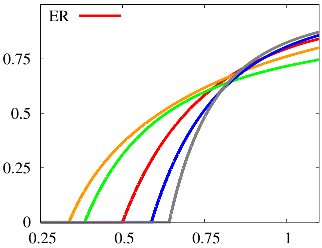
There is rule-dependent $c > 0$ such that for *constant* $\varepsilon > 0$, whp,

$$L_1(t_c n + \varepsilon n) \approx c\varepsilon n$$

Some further developments

- Generalized Bohman–Frieze rules (Drmota–Kang–Panagiotou)
- Critical window (Bhamidi–Budhiraja–Wang)
- Other properties (Kang–Perkins–Spencer and Sen)

New Results for Bounded-Size Rules (1/4)



Linear growth of the giant component (Riordan–W.)

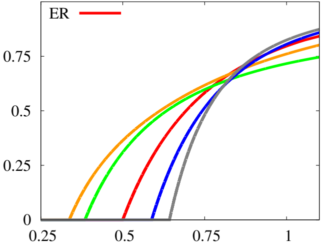
For *any* bounded-size rule there is $c > 0$ such that for $\varepsilon \gg n^{-1/3}$, whp,

$$L_1(t_c n + \varepsilon n) \approx c\varepsilon n$$

Remarks

- Same qualitative behaviour as in Erdős–Rényi process
- Previous results: for *constant* $\varepsilon > 0$ and *restricted* class of rules

New Results for Bounded-Size Rules (1/4)



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For *any* bounded-size rule there is $c > 0$ such that for $\varepsilon \gg n^{-1/3}$, whp,

$$L_1(t_c n + \varepsilon n) \approx c\varepsilon n$$

Remarks

- We also obtain whp $L_1(t_c n - \varepsilon n) \approx C\varepsilon^{-2} \log(\varepsilon^3 n)$
- Our L_1–results establish a number of conjectures (Janson–Spencer, Borgs–Spencer, Kang–Perkins–Spencer, Bhamidi–Budhiraja–Wang)

New Results for Bounded-Size Rules (2/4)

Size of the largest subcritical component (Riordan–W.)

For *any* bounded-size rule there is $C > 0$ such that for $\varepsilon \gg n^{-1/3}$, whp,

$$L_1(t_c n - \varepsilon n) \approx C\varepsilon^{-2} \log(\varepsilon^3 n)$$

Remarks

- Same *qualitative form* as in Erdős–Rényi process
- Conjectured by Kang–Perkins–Spencer and Bhamidi–Budhiraja–Wang
- Improves results of Bhamidi–Budhiraja–Wang and Sen

New Results for Bounded-Size Rules (3/4)

Number of vertices in small components (Riordan–W.)

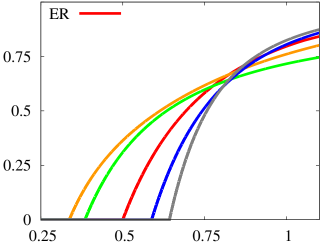
For *any* bounded-size rule: as $k \to \infty$ and $\varepsilon \to 0$, we have

$$N_k(t_c n \pm \varepsilon n) \approx C k^{-3/2} e^{-(c+o(1))\varepsilon^2 k} n$$

Remarks

- Same *qualitative form* as in Erdős–Rényi process
- Conjectured by Kang–Perkins–Spencer and Drmota–Kang–Panagiotou
- Improves partial results of Drmota–Kang–Panagiotou

New Results for Bounded-Size Rules (4/4)



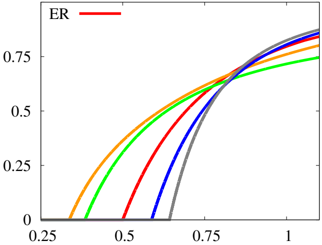
Take-home message (universality)

Phase transition of all bounded-size rules exhibits Erdős–Rényi behaviour

For example, for rule-dependent constants $t_c, c, C > 0$ we whp have

$$L_1(i) \approx \begin{cases} C\varepsilon^{-2}\log(\varepsilon^3 n) & \text{if } i = t_c n - \varepsilon n, \\ c\varepsilon n & \text{if } i = t_c n + \varepsilon n, \end{cases}$$

New Results for Bounded-Size Rules (4/4)



Take-home message (universality)

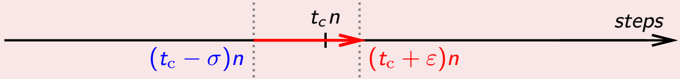
Phase transition of all bounded-size rules exhibits Erdős–Rényi behaviour

The 120+ pages proof uses a blend of techniques, including

- Combinatorial two-round exposure arguments,
- Differential equation method,
- PDE theory,
- Branching processes, . . .

Structure of the Proof

Focus on evolution around critical point



Proof strategy

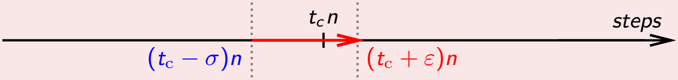
- Track bounded-size rule only up to step $(t_c - \sigma)n$
- Go from $(t_c - \sigma)n$ to $(t_c + \varepsilon)n$ via *two-round exposure*
- Analyze component-size distribution via *branching-process*

In comparison with previous approaches

- We track the process directly (no approximation)
- We can allow for $\varepsilon = \varepsilon(n) \to 0$

Structure of the Proof

Focus on evolution around critical point



Proof strategy

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Exemplar techniques

- Differential equation method + exploration arguments
- Branching processes + large deviation arguments

Glimpse of the Proof (1/2)

Preprocessing graph after $(t_c - \sigma)n$ steps:

- S contains all vertices in components with size $\leq B$
- L contains all other vertices (i.e., with component-sizes $> B$)

First exposure of all steps $(t_c - \sigma)n, \ldots, (t_c + \varepsilon)n$

- reveal which vertices of $(v_1, \ldots, v_4)$ are in S or L
- for those v_j in S, also reveal *which vertex* of S

Crucial observation

Enough to inductively make all decisions (whether edge e_1 or e_2 added)

Proof: inductively track

- edges added to S
- edges connecting S to L (their endvertices in S)

Glimpse of the Proof (2/2)

Knowledge after first exposure round

- S: component structure (incl. number of incident S–L edges)
- L: component structure + total number of (random) L–L edges

Key observation

- So-far undetermined L-vertices are all *uniformly* distributed

Simple description of second exposure round

- for each S–L edge: pick random endvertex in L
- add prescribed number of purely random L–L edges

$\Rightarrow$ **Can explore resulting graph via branching process**

Some Difficulties

Some difficulties

- very little 'explicit' knowledge about the variables/functions involved
- approximation errors are everywhere (e.g., random fluctuations)

Bootstrapping knowledge about $X_k(tn) \approx x_k(t)n$

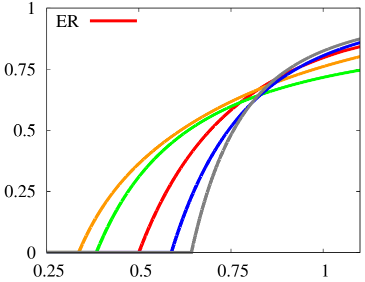
- *Differential equation method*: $x_k(t)$ solves differential equations
- *Branching process based approach*: $x_k(t) \leq Ae^{-ak}$
- Combining both (analyzing combinatorial structure of x'_k):

$$x_k^{(j)}(t) \leq B_j e^{-bk}$$

Summary

Phase transition of bounded-size rules

Same qualitative behaviour as in Erdős–Rényi process



Open problem

How can we analyze 'unbounded' size rules (e.g., the sum rule)?