The evolution of subcritical Achlioptas processes

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Joint work with Oliver Riordan

The model

Achlioptas processes

- Start with an empty graph on n vertices
- In each step: pick *two* random edges,
  add *one* of them to the graph (using some *rule*)

Remarks

- Yields *family* of random graph processes
- Contains 'classical' Erdős–Rényi process

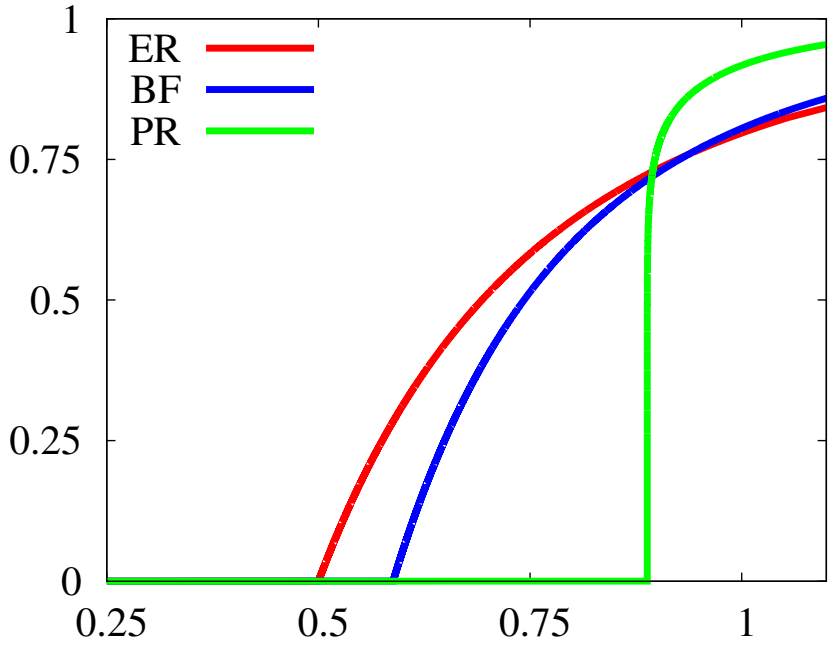
Motivation

- Improve our understanding of phase transition phenomena
- Test / develop methods for analyzing processes with dependencies

Behaviour of different Achlioptas proceses

Key example (suggested by Achlioptas)

Fraction of vertices in largest component after tn steps: $L_1(tn)/n$

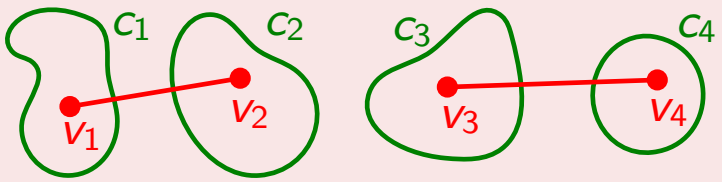


Goal of this talk

Understand how these evolve over time.

Widely studied Achlioptas rules

Size rules



Decision (which edge to add) depends *only* on component *sizes* $c_1, \ldots, c_4$

- Product rule (Bollobás)

Bounded-size rules (most of previous work)

All component sizes larger than some constant B are treated the same

- Erdős–Rényi ($B = 0$)
- Bohman–Frieze ($B = 1$)

Previous work

Bounded-size rules (Spencer–Wormald, Bohman–Kravitz, Riordan–W.,. . .)

For any bounded-size rule $\mathcal{R}$ the key statistics are convergent:

- *Small components*: $N_k(tn) \sim \varrho_k^{\mathcal{R}}(t)n$
- *Largest component*: $L_1(tn) \sim \varrho^{\mathcal{R}}(t)n$

- Proofs use Wormald's differential equation method
- $\varrho_k^{\mathcal{R}}$ determined by an associated system of DEs

Size rules (Riordan–W.)

For any 'sensible' size rule $\mathcal{R}$ the key statistics are convergent
IF an associated system of differential equations has a *unique* solution.

- 'Direct' proof
- Caveat: unique solution
  - Well-known for 'nice' size rules (e.g., bounded-size rules)
  - Open problem for general size rules (e.g., product rule)

New Result for Size Rules

Susceptibility $\chi(G) = \frac{1}{n}\sum_{k\geq 1} kN_k(G)$

- Expected size of component containing randomly selected vertex

Riordan–W. (simplified)

Any size rule $\mathcal{R}$ is 'well-behaved' until the critical time $t_c = t_c^{\mathcal{R}}$, where the susceptibility χ diverges. For $t < t_c$ whp

- *Small components*: $N_k(tn) \sim \varrho_k^{\mathcal{R}}(t)n$
- *Exponential tails*: $N_k(tn) \leq Ae^{-ak}n$
- *Largest component*: $L_1(tn) \leq B\log n$

Remarks

- Removes earlier uniqueness assumption up to t_c
- t_c is important time in evolution

Critical time t_c

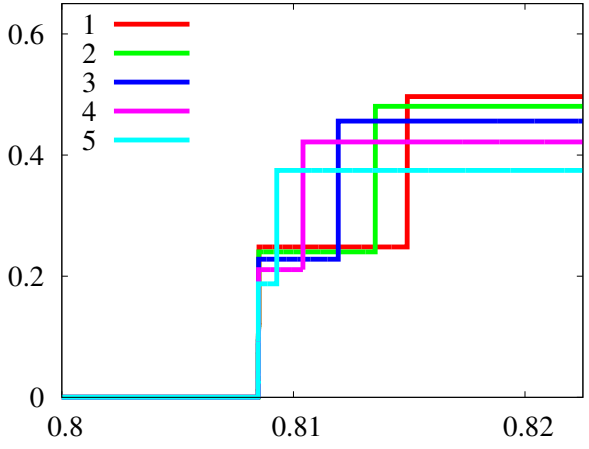
Conjecture for size rules (simplified)

For $t > t_c$ we have a giant component: $L_1(tn) = \Omega(n)$

- Was known for bounded-size rules (Spencer–Wormald)
- True for certain classes of size rules (e.g., maximum product rule)

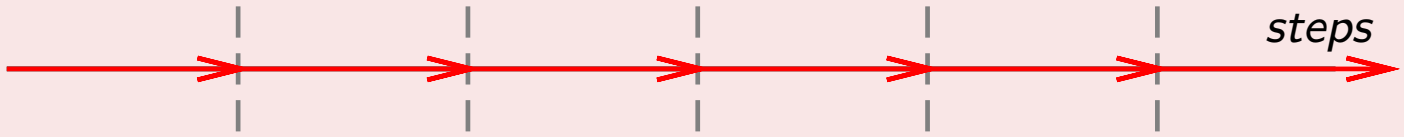
Convergence up to t_c seems best possible

Beyond t_c some rules look *nonconvergent* in simulations



Structure of the proof

Inductively establish concentration



Need: evolution starting from initial graph F

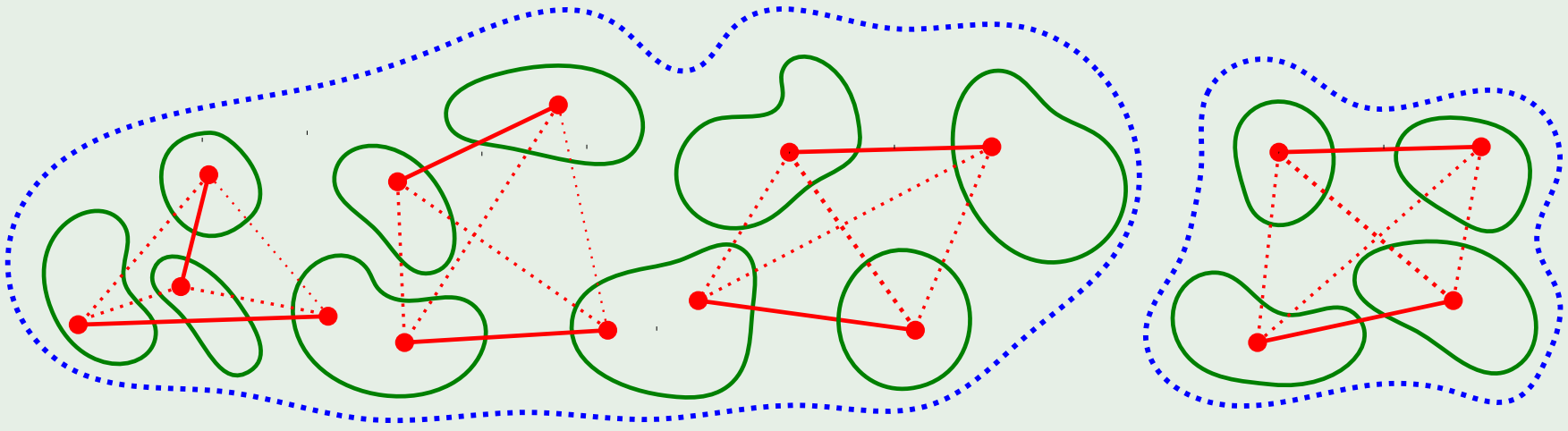
- Assumption: initial graph F is 'nice'
- Conclusion: graph after σn steps is again 'nice' (if σ *small enough*)

In comparison to bounded size rules

- We track key statistics *without* using differential equations
- We *investigate dependencies* amoung choices in more detail

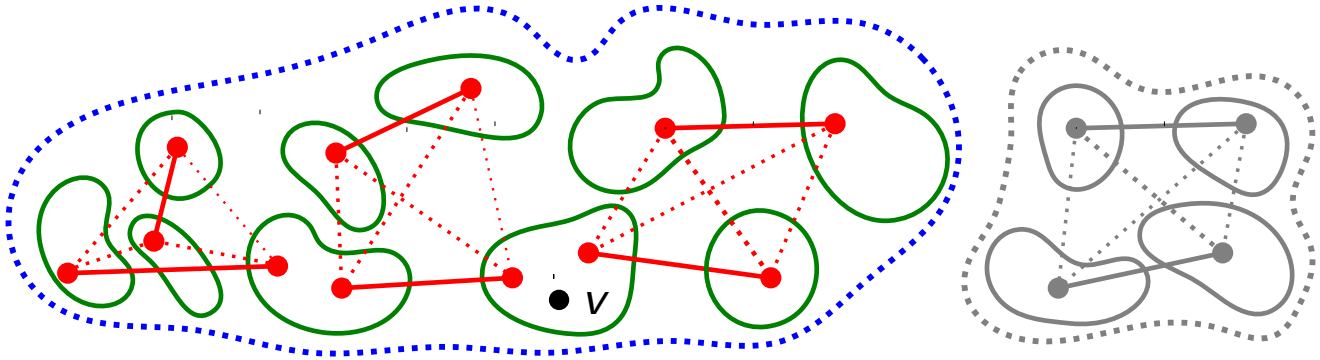
Investigating dependencies

How far can decisions propagate?



- For size rules, decisions can only propagate inside clusters
  - Here we *ignore order* of pairs
- Inside each cluster:
  - *Order of the pairs* uniquely determines decisions of any size rule

Glimpse of the proof



Determine component size $|C_v|$ via two-step exposure

- Reveal *all pairs of edges* offered
  - Determine relevant cluster for v $\approx$ Branching process
- Reveal *order* of all (relevant) pairs
  - Apply size rule $\mathcal{R}$ inside cluster

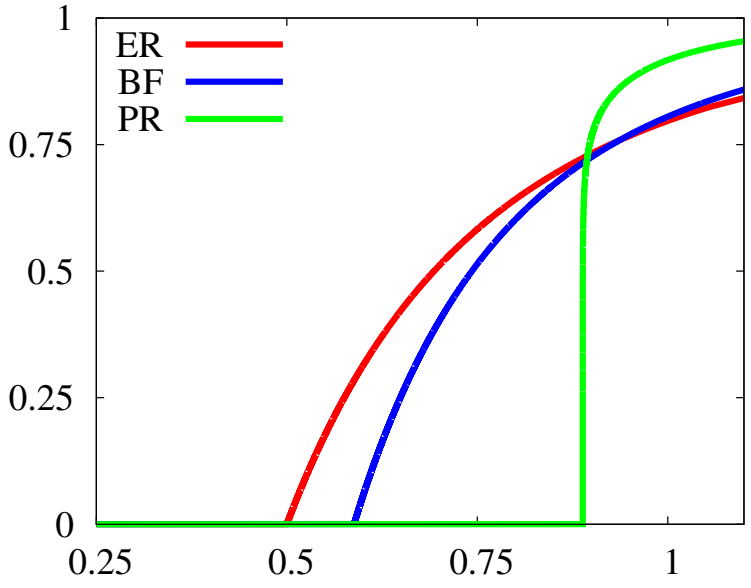
Why do we need susceptibility $\chi < \infty$?

- Branching process must be 'sub-critical' (need $\sigma \leq c\chi^{-1}$)
  - Only 'few' edges/components influence $|C_v|$ $\rightarrow$ Concentration

Summary

First rigorous result for size rules (Riordan–W.)

Key statistics are 'well-behaved' until the susceptibility χ diverges.



Open problem

How can we analyze the later evolution of size rules?