Convergence of Achlioptas processes via differential equations with unique solutions

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Joint work with Oliver Riordan

Achlioptas process

Achlioptas processes

- Start with an empty graph on n vertices
- Repeatedly: pick *two* random edges,
  using some *rule*, add *one* of them to the graph

Remarks:

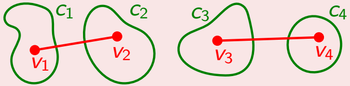
- Yields *family* of random graph processes
- Contains 'classical' Erdős-Rényi model

This talk:

- Want to understand how these evolve over time

Widely studied Achlioptas rules

Size rules



Decision (which edge to add) depends *only* on component *sizes* $c_1, \ldots, c_4$.

Examples

- Sum and product rules (minimize sum/product of c_i)

Bounded-size rules

All component sizes larger than some constant B are treated the same.

Examples

- Erdős-Rényi rule ($B = 0$)
- Bohman-Frieze rule ($B = 1$)

Previous Work

Bounded-size rules (Bohman-Kravitz, Spencer-Wormald, Riordan-W.,...)

Undergo phase transition. Key statistics are *concentrated* and *convergent*:

- Vertices in 'small' components: $N_k(tn) \approx \varrho_k^{\mathcal{R}}(t)n$
- Largest component: $L_1(tn) \approx \varrho^{\mathcal{R}}(t)n$

Key points

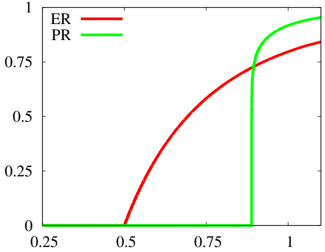
- Qualitatively *similar* to Erdős-Rényi case
- Proofs use Wormald's 'differential equation method'

Size rules

Very few rigorous results; maybe *different* behaviour?

A cautionary Tale

Simulation of $\varrho(t) = \frac{L_1(tn)}{n}$ for the product rule (PR)



Conjecture of Achlioptas-D'Souza-Spencer (*Science* 2009)

The product rule exhibits a *discontinuous* phase transition.

Riordan-W. (*Science* 2011)

For *all* Achlioptas processes the phase transition is *continuous*.

Main Result

Simplified main result for size rules (Riordan-W. 2011+)

If a natural system of differential equations has a *unique* solution we establish *concentration* and *convergence*:

- Vertices in 'small' components: $N_k(tn) \approx \varrho_k^{\mathcal{R}}(t)n$
- Largest component: $L_1(tn) \approx \varrho^{\mathcal{R}}(t)n$

Remarks

- Generalizes previous results in the area
  - For these uniqueness is well-known
- System of differential equations may be infinite (depends on rule $\mathcal{R}$)

Main contribution

New approach for proving convergence to solution of differential equations

Differential Equations Method

Differential Equations Method (Wormald, 1995)

Under suitable technical conditions

$$\mathbb{E}[X_{tn+1} - X_{tn} \mid \mathcal{F}_{tn}] \approx f(t, X_{tn}/n),$$

implies that we obtain concentration/convergence

$$X_{tn}/n \approx x(t),$$

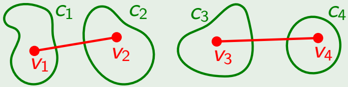
where $x(t)$ is unique solution to $x'(t) = f(t, x(t))$.

Remarks

- Plausible as $X_{tn} \approx x(t)n$ suggests $X_{tn+1} - X_{tn} \approx x'(t)$
- Technical conditions imply uniqueness

The System of Differential Equations

Size rules decide using $c_1, \ldots, c_4$ only



$d_k(c_1, \ldots, c_4)$ = change of N_k given component sizes $c_1, \ldots, c_4$

Simplification: let's assume $v_1, \ldots, v_4$ are in *different* components

System of differential equations

Motivated by *expected one-step change* of $N_k(tn) \approx \varrho_k(t)n$:

$$\varrho_k'(t) = \sum_{c_1,\ldots,c_4 \in \mathbb{N}\cup\{\infty\}} d_k(c_1, \ldots, c_4) \prod_{j\in[4]} \varrho_{c_j}(t)$$

Key Idea of Proof

Main steps for showing $N_k(tn) \approx \varrho_k(t)n$

- We start with an infinite set of sample points

$$\omega_n \in \Omega_n$$

- Pick subsequence (ω_n) such that for every $t \geq 0$ and $k \geq 1$:

$$\frac{N_k(tn)(\omega_n)}{n} \to \varrho_k(t)$$

- Show that the $\varrho_k(t)$ satisfy the system of differential equations:

$$\varrho_k'(t) = F_k(t, \varrho_1, \varrho_2, \ldots)$$

- If this has a unique solution $(\hat{\varrho}_k)_{k\geq 1}$ then

$$\varrho_k = \hat{\varrho}_k,$$

so ϱ_k does *not* depend on the selected subsequence!

Summary

Simplified main result (Riordan-W.)

Achlioptas processes using size rules are *concentrated* and *convergent* if an associated system of differential equations has a *unique* solution.

Main contribution

- New approach for proving convergence to solution of DE

Open problem

Can we establish uniqueness for the product rule?

Simplified result for size rules (Riordan–W. 2011+)

If a natural system of differential equations has a *unique* solution, then we establish *convergence* of key statistics:

- 'Small' components: $\frac{N_k(tn)}{n} \xrightarrow{\mathrm{p}} \varrho_k^{\mathcal{R}}(t)$
- Largest component: $\frac{L_1(tn)}{n} \xrightarrow{\mathrm{p}} \varrho^{\mathcal{R}}(t)$

Remarks

- Generalizes previous results in the area (bounded size rules)
  - For these uniqueness is easy/well-known
- System of differential equations may be infinite (depends on rule $\mathcal{R}$)
  - Causes technical difficulties (for, e.g., product rule)
  - Can remove uniqueness assumption up to certain critical t_{c}