A deletion method for local subgraph counts

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Random Graphs

Random graph $G_{n,p}$

- n vertices
- each of the $\binom{n}{2}$ edges appears independently with probability p

Small subgraph H

- graph of *fixed* size (v vertices and e edges)
- $X_H =$ number of H-subgraphs in $G_{n,p}$

Expected number of H-subgraphs in $G_{n,p}$

- $\mathbb{E}[X_H] = \Theta(n^v p^e)$

Is the number of H-subgraphs close to its expectation?

Small subgraphs in Random Graphs

X_H = number of H-subgraphs in $G_{n,p}$

Is the number of H-subgraphs close to its expectation?

In many applications we want $X_H \approx \mathbb{E}[X_H]$

Error probability:

'very small' $\approx 2^{-\Theta(\mathbb{E}[X_H])}$

'small' $\approx 2^{-\Theta(\sqrt{\mathbb{E}[X_H]})}$

How concentrated is X_H around $\mathbb{E}[X_H]$?

Fact: Number of H-subgraphs $\approx \mathbb{E}[X_H]$ (Janson, Kim, Vu, ...)

The number of H-subgraphs is close to its expectation:

$$\mathbb{P}[X_H \leq (1-\varepsilon)\mathbb{E}[X_H]] = \text{`very small'}$$
$$\mathbb{P}[X_H \geq (1+\varepsilon)\mathbb{E}[X_H]] = \text{`small'}$$

Heuristic reason for asymmetry:

- can create ‘many’ H-copies by adding comparatively ‘few’ edges
- by deleting ‘few’ edges we can’t always delete ‘many’ H-copies

Deleting a few edges might help?

Deletion Method

'Deletion Lemma' (Rödl-Ruciński, 1995)

With 'very high' probability it suffices to delete a 'few edges' to ensure that the remaining graph does not contain 'too many' copies of H, i.e.

$$X_H \leq (1+\varepsilon)\mathbb{E}[X_H]$$

Usually applied together with a 'Robustness-Lemma'

- deleting a 'few' edges does not destroy too many copies of H

'Deletion Lemma'+'Robustness-Lemma' (Rödl-Ruciński, 1995)

With 'very high' probability it suffices to delete a 'few edges' to ensure that the remaining graph contains the 'correct' number of copies of H, i.e.

$$(1-\varepsilon)\mathbb{E}[X_H] \leq X_H \leq (1+\varepsilon)\mathbb{E}[X_H]$$

Bounded 'local' Subgraph Counts

Sometimes global bound on number of H-subgraphs is not enough!

In applications 'local' bounds are useful

- bounds on the number of H-copies per edge/vertex

In the following we focus on triangles

- strengthening of the 'Deletion Lemma' of Rödl-Ruciński
- obtain 'local' bound on the number of triangles (per edge/vertex)

'Local Triangle Deletion Lemma'

'Local Triangle Deletion Lemma' (Spöhel-Steger-W., 2009+)

With 'very high' probability we can delete a 'few' edges such that in the remaining graph:

- the global triangle-count is 'correct'
- the 'local' triangle-count (per vertex/edge) is 'bounded'

Notation

- X_Δ = number of triangles

Global triangle-count is 'correct'

- $(1-\varepsilon)\mathbb{E}[X_\Delta] \leq X_\Delta \leq (1+\varepsilon)\mathbb{E}[X_\Delta]$

'Local Triangle Deletion Lemma'

'Local Triangle Deletion Lemma' (Spöhel-Steger-W., 2009+)

With 'very high' probability we can delete a 'few' edges such that in the remaining graph:

- the global triangle-count is 'correct'
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'Few' edges

- at most $\varepsilon \min\left\{\binom{n}{2}p, \mathbb{E}[X_\Delta]\right\}$ many

Why not $\varepsilon\binom{n}{2}p$ many edges?

- then we could delete all triangles for $\mathbb{E}[X_\Delta] \ll \binom{n}{2}p$

'Local Triangle Deletion Lemma'

'Local Triangle Deletion Lemma' (Spöhel-Steger-W., 2009+)

With 'very high' probability we can delete a 'few' edges such that in the remaining graph:

- the global triangle-count is 'correct'
- the 'local' triangle-count (per vertex/edge) is 'bounded'

Notation

- X_v = number of triangles per vertex v

Triangle-count per vertex is 'bounded'

- $X_v \leq \max\{C, (1+\varepsilon)\mathbb{E}[X_v]\}$

Why not $X_v \leq (1+\varepsilon)\mathbb{E}[X_v]$?

- for certain p: $X_\Delta \geq 1$ and $\mathbb{E}[X_v] \to 0$

‘Local Triangle Deletion Lemma’

‘Local Triangle Deletion Lemma’ (Spöhel-Steger-W., 2009+)

With ‘very high’ probability we can delete a ‘few’ edges such that in the remaining graph:

- the global triangle-count is ‘correct’
- the ‘local’ triangle-count (per vertex/edge) is ‘bounded’

Notation

- X_e = number of triangles per edge e

Triangle-count per edge is ‘bounded’

- $X_e \leq \max\{C, (1+\varepsilon)\mathbb{E}[X_e]\}$

Why not $X_e \leq (1+\varepsilon)\mathbb{E}[X_e]$?

- for certain p: $X_\Delta \geq 1$ and $\mathbb{E}[X_e] \to 0$

‘Local Triangle Deletion Lemma’

‘Local Triangle Deletion Lemma’ (Spöhel-Steger-W., 2009+)

With ‘very high’ probability we can delete at most $\varepsilon \min \left\{ \binom{n}{2} p, \mathbb{E}[X_\Delta] \right\}$ edges such that in the remaining graph:

- the global triangle-count is ‘correct’:
  - $(1-\varepsilon)\mathbb{E}[X_\Delta] \leq X_\Delta \leq (1+\varepsilon)\mathbb{E}[X_\Delta]$
- the ‘local’ triangle-count is ‘bounded’:
  - $X_v \leq \max\{C, (1+\varepsilon)\mathbb{E}[X_v]\}$
  - $X_e \leq \max\{C, (1+\varepsilon)\mathbb{E}[X_e]\}$

Strengthening of Rödl-Ruciński ‘Deletion Lemma’ for triangles:

- only guarantees that the global triangle-count is ‘correct’

Key-Lemma of the Proof

Key Lemma

With 'very high' probability there exists a subgraph with:

- reasonable 'many' triangles
- every vertex/edge is not contained in 'too many' triangles

Main ingredient of the proof:

- an application of the so-called FKG Inequality

Monotone Graph-Property $\mathcal{P}$

$\mathcal{P}$ increasing $\Leftrightarrow$ it can't be destroyed by adding edges
$\mathcal{P}$ decreasing $\Leftrightarrow$ it can't be destroyed by deleting edges

Examples:

- connectivity: increasing
- k-colorability: decreasing

Observation:

- $\mathcal{P}$ increasing $\iff$ $\neg\mathcal{P}$ decreasing

FKG Inequality

FKG Inequality (Fourtain-Kasteleyn-Ginibre, 1971)

Let $\mathcal{A}$ and $\mathcal{B}$ be two decreasing graph properties. Then for $G_{n,p}$ we have

$$\mathbb{P}[\mathcal{A}] \leq \mathbb{P}[\mathcal{A} \mid \mathcal{B}]$$

i.e. the probability of a decreasing event $\mathcal{A}$ does not decrease if we condition on another decreasing event $\mathcal{B}$

Example:

- $\mathcal{A} =$ being k-colorable
- $\mathcal{B} =$ maxdegree at most $k + 2$

FKG Inequality (Fourtain-Kasteleyn-Ginibre, 1971)

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$$\mathbb{P}[\mathcal{A}] \leq \mathbb{P}[\mathcal{A} \mid \mathcal{B}]$$

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Remarks:

- statement also holds for two increasing events $\mathcal{A}$ and $\mathcal{B}$
- not valid for arbitrary probability spaces
  - in particular not for the random graph $G_{n,m}$

FKG Trick

Events

- $\mathcal{S}$ = there exists a subgraph satisfying $\mathcal{I}$ and $\mathcal{D}$
- $\mathcal{I}$ = increasing Property
- $\mathcal{D}$ = decreasing Property

Observations

- $\mathcal{S}$ is increasing $\iff$ $\neg\mathcal{S}$ is decreasing
- $\neg\mathcal{S} \cap \mathcal{D}$ implies $\neg\mathcal{I} \implies \mathbb{P}[\neg\mathcal{S} \cap \mathcal{D}] \leq \mathbb{P}[\neg\mathcal{I}]$

FKG Trick

$$\mathbb{P}[\neg\mathcal{S}] \leq \mathbb{P}[\neg\mathcal{S} \mid \mathcal{D}] = \frac{\mathbb{P}[\neg\mathcal{S} \cap \mathcal{D}]}{\mathbb{P}[\mathcal{D}]} \leq \frac{\mathbb{P}[\neg\mathcal{I}]}{\mathbb{P}[\mathcal{D}]}$$

$\Rightarrow$ we reduced the problem of bounding $\mathbb{P}[\neg\mathcal{S}]$ to bounding $\mathbb{P}[\neg\mathcal{I}]$ from above and $\mathbb{P}[\neg\mathcal{D}]$ from below

Proof of Key Lemma using FKG Trick

Key Lemma (Simplified)

With 'very high' probability there exists a subgraph such that:

- there are 'many' triangles $(\mathcal{I})$
- every vertex/edge is not contained in 'too many' triangles $(\mathcal{D})$

Define Events

- $\mathcal{S}$ = there exists a subgraph satisfying $\mathcal{I}$ and $\mathcal{D}$
- monotonicity: $\mathcal{I}$ increasing and $\mathcal{D}$ decreasing

FKG Trick implies

$$\mathbb{P}[\neg\mathcal{S}] \leq \frac{\mathbb{P}[\neg\mathcal{I}]}{\mathbb{P}[\mathcal{D}]} \leq 2\,\mathbb{P}[\neg\mathcal{I}] = \text{'very small'}$$

Technical Lemma

$$\mathbb{P}[\neg\mathcal{I}] = \text{'very small'} \qquad \text{and} \qquad \mathbb{P}[\mathcal{D}] \geq 1/2$$

Summary

'Local Triangle Deletion Lemma' (Spöhel-Steger-W., 2009+)

with 'very high' probability:

deleting a few edges $\Longrightarrow$ fix global + bound local triangle counts

Strengthening of the Rödl-Ruciński 'Deletion Lemma' for triangles:

'Deletion Lemma' (Rödl-Ruciński, 1995)

with 'very high' probability:

deleting a few edges $\Longrightarrow$ fix global subgraph count

Work in progress:

- extension to general case (arbitrary subgraphs)