Dense subgraphs in the H-free process

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Random graph processes

Random graph process

(a) Start with empty graph on n vertices

(b) Add edges, one at a time, chosen uniformly at random from all remaining pairs.

Random H-free graph process

(a) Start with empty graph on n vertices

(b) Add edges, one at a time, chosen uniformly at random from all remaining pairs *that do not complete a copy of* H.

Basic questions:

(1) Final number of edges? (Erdős-Suen-Winkler, 1995)

(2) Subgraph counts?

Properties of the H-free process

In this talk H satisfies some 'density condition'

Final number of edges

- known for the K_3-free process up to constants (Bohman, 2009)
- known for the H-free process only up to log-factors (Osthus-Taraz, 2001)

Subgraph counts

- Comparable to normal random graph during the first m steps, where

$$m \approx \delta\, n^{2-1/d_2(H)}\, (\log n)^{c(H)}$$

  (Bohman-Keevash, 2009+)

- Behaviour in later steps remains open

Recap: Small subgraphs in $G(n,i)$

Maximum Density

For a graph F it is determined by 'densest subgraph':

$$m(F) := \max_{J \subseteq F, e_F \geq 1} \left\{ \frac{e_J}{v_J} \right\} \quad .$$

Small subgraphs theorem (Bollobás)

Suppose we have a fixed graph F and $i = n^{2-1/\alpha}$.
Then we have a 'threshold phenomenon' in $G(n,i)$:

$$\text{whp} \begin{cases} \text{no copy of } F & \text{if } m(F) > \alpha \\ \text{'many' copies of } F & \text{if } m(F) < \alpha \end{cases}$$

Results of Bohman-Keevash

Fixed subgraphs in the H-free process

Bohman-Keevash showed that for fixed F with $H \not\subseteq F$, in the graph produced by the H-free process after the first $m = \delta n^{2-1/d_2(H)}(\log n)^{c(H)}$ steps:

$$\text{whp} \begin{cases} \text{no copy of } F & \text{if } m(F) > d_2(H) \\ \text{`many' copies of } F & \text{if } m(F) < d_2(H) \end{cases}$$

$\Longrightarrow$ **H-free process 'looks' almost like a normal random graph, but it has no copies of H!**

What happens in later steps?

- can 'very dense' subgraphs appear?

Previous results

K_3-free process (Gerke-Makai, 2010+)

There exists $c > 0$ such that whp any fixed F with

$$m(F) \geq c$$

does not appear in the K_3-free process.

$\Longrightarrow$ **No 'very dense' subgraphs in later steps!**

What happens in the H-free process?

- what about graphs with $v_F = \omega(1)$ vertices?

Our result

H-free process (W., 2010+)

There exists $c(H), d(H) > 0$, such that the H-free process has whp no subgraph J with density

$$m(J) \geq c(H)$$

on $v_J \leq n^{d(H)}$ vertices.

$\Longrightarrow$ **No 'very dense' subgraphs in later steps, even if they are 'large' ('many' vertices)!**

Remarks

- extends/generalizes results for K_3-free process
- tight up to the constant:
  - whp fixed F with $m(F) < d_2(H)$ appear

Proof idea

Goal:

- whp no copy of J appears in the H-free process

Main idea

- We prove that whp already after the first m steps:
  - for *every* possible placement of J, at least one of its pairs is 'closed' (i.e. can not be added in later steps)

Proof Strategy

- Show that whp in each step there are 'many' pairs that would close at least one pair of J
- Avoiding those pairs in *all* m steps is 'very unlikely'

Summary

For H that satisfies some 'density condition':

The H-free process contains no 'very dense' subgraphs

- Whp the H-free process has no subgraphs J on $v_J \leq n^{d(H)}$ vertices with density $m(J) \geq c(H)$

Conjecture (W., 2010+)

- The H-free process contains whp no copy of a fixed graph F with $m(F) > d_2(H)$.