Packing nearly optimal Ramsey $R(3, t)$ graphs

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Joint work with He Guo

Context of this talk

Ramsey number $R(s,t)$

$R(s,t) :=$ minimum $n \in \mathbb{N}$ such that every red/blue edge-coloring of complete n-vertex graph K_n contains red K_s or blue K_t

- Major problem in combinatorics: determining asymptotics
- Testbed for new proof techniques/methods:
  Alteration, LLL, Concentration Ineq., Semi-Random, Differential Eq.

Celebrated Result (Ajtai-Komlós-Szemerédi 1980 + Kim 1995)

$R(3,t) = \Theta(t^2/\log t)$

- Lower bound harder: Kim received Fulkerson Prize 1997
- $R(3,t) = \Omega(t^2/(\log t)^2)$ already by Erdős in 1961

Topic of this talk

Extension of Kim-result (implies asymptotics of other Ramsey parameter)

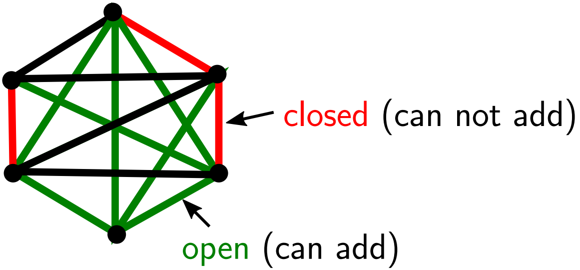
Main Result: nearly optimal $R(3,t)$ graphs

Kim (1995) + Bohman (2008): one nearly optimal $R(3,t)$ graph

Both find an n-vertex graph $G \subseteq K_n$ such that
G is Δ-free with independence number $\alpha(G) \le C\sqrt{n \log n}$

- Using (semi-random variation of) Δ-*free process*: greedily add random edges that do not close a Δ

Δ-free process: add one random edge in each step



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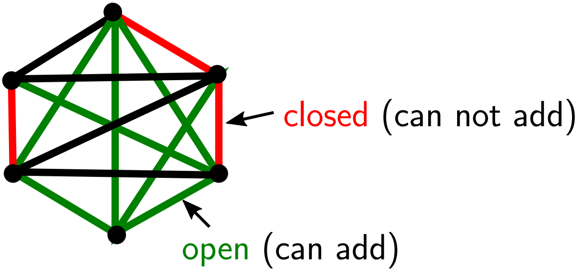
Semi-random variation: add many random-like edges in each step



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Guo, W. (2017+): almost packing of nearly optimal $R(3,t)$ graphs

Given $\varepsilon > 0$, we find edge-disjoint graphs $(G_i)_{i \in \mathcal{I}}$ with $G_i \subseteq K_n$ such that
(a) each G_i is Δ-free with $\alpha(G_i) \leq C_\varepsilon\sqrt{n \log n}$
(b) the union of the G_i contains $\geq (1-\varepsilon)\binom{n}{2}$ edges

- Using simple *polynomial-time randomized algorithm*:
  sequentially choose G_i via semi-random variation of Δ-free process

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Motivation: why should we care?

- Natural packing extension of Kim's result
- Technical challenge: controlling errors over $\Theta(\sqrt{n / \log n})$ iterations
- Establishes Ramsey-Theory conjecture by Fox et.al. (cf. next slides)

Ramsey Theory with $r \geq 2$ colors

$G \to (H)_r$ $:\Leftrightarrow$ any r-coloring of $E(G)$ has monochromatic copy of H

Ramsey theory $\triangleq$ studying properties of "r-Ramsey minimal graphs"

$\mathcal{M}_r(H) :=$ all graphs G that are r-Ramsey minimal for H
(i.e., $G \to (H)_r$ and $G' \not\to (H)_r$ for all $G' \subsetneq G$)

- $\min_{G \in \mathcal{M}_r(K_k)} v(G) =$ Ramsey number
- $\min_{G \in \mathcal{M}_r(K_k)} e(G) =$ Size Ramsey number

Minimum degree of r-Ramsey minimal graphs (Burr, Erdős, Lovász 1976)

$s_r(H) := \min_{G \in \mathcal{M}_r(H)} \delta(G)$

- $s_2(K_k) = (k-1)^2$: Burr, Erdős, Lovász (1976)
- $s_2(H) = 2\delta(H) - 1$: for many bipartite H (trees, $K_{a,b}$, etc)
  Fox, Lin (2006) + Szabó, Zumstein, Zürcher (2010)
- $s_r(K_k) = \tilde{\Theta}_k(r^2)$: Fox, Grinshpun, Liebenau, Person, Szabó (2015)

Ramsey Conjecture of Fox et.al.

Minimum degree of minimal r-Ramsey graphs (Burr, Erdős, Lovász 1976)

$s_r(K_k) := min_{G \in \mathcal{M}_r(K_k)} \delta(G)$

- $cr^2 \log r \leq s_r(K_3) \leq Cr^2(\log r)^2$ by FGLPS (2015)

Conjecture (Fox, Grinshpun, Liebenau, Person, Szabo, 2015)

$s_r(K_3) = O(r^2 \log r)$

- They suggested to pack G_i sequentially via Δ-free process (their weaker upper bound relies on sequential LLL–argument)

Conj. True (Guo, W. 2017+): corollary of our main packing result

Implies $s_r(K_3) = \Theta(r^2 \log r)$

- For technical reasons: use *semi-random variation* of Δ-free process

Glimpse of the proof

Main-Technical-Result: find random-like Δ-free subgraph $G \subseteq H$

Let $\varrho := \sqrt{\beta(\log n)/n}$ and $s := C_\varepsilon\sqrt{n \log n}$. If $H \subseteq K_n$ is such that

$$e_H(A, B) \geq \varepsilon \, |A||B|$$

for all disjoint sets A, B of size s, then we can find Δ-free $G \subseteq H$ with

$$e_G(A, B) = (1 \pm \delta) \, \varrho \, e_H(A, B)$$

for all disjoint A, B of size s.

Proof based on semi-random variation of Δ-free process:

- Do *not* require degree/codegree regularity of H
- 'Self-stabilization' mechanism built into process (to control errors)
- Tools: Bounded-Differences-Ineq. and Upper-Tail-Ineq. of mine

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Implies packing result: *(maintaining $e_{H_i}(A, B)$ bounds inductively)*

- Start with $H_0 = K_n$
- Sequentially choose $G_i \subseteq H_i$ and set $H_{i+1} = H_i \setminus G_i$
- Stop when $e_{H_I}(A, B) \approx \varepsilon |A||B|$ holds

Semi-random construction of Δ-free subgraph

To construct triangle-free T_J, we iteratively keep track of

- E_j: "random" set of edges
- $T_j \subseteq E_j$: Δ-free and $|T_j| \approx |E_j|$
- $O_j \subseteq \{\text{all } e \notin E_j \text{ that don't form a } \Delta \text{ with any two edges of } E_j\}$

Idea of each step

(1) Generate few random edges $\Gamma_{j+1} \subseteq O_j$
(2) Alteration: find $\Gamma'_{j+1} \subseteq \Gamma_{j+1}$ s.t. $T_{j+1} = T_j \cup \Gamma'_{j+1}$ remains Δ-free
(3) Update $O_{j+1} \subseteq O_j \setminus \Gamma_{j+1}$

Random edge-set Γ_{i+1} and edge-set E_{j+1}

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Definition of Γ_{j+1} and E_{j+1}

- $\Gamma_{j+1} \subseteq O_j$: p-random subset of O_j
- $E_{j+1} = E_j \cup \Gamma_{j+1}$

Why can we ensure $|\Gamma'_{j+1}| \approx |\Gamma_{j+1}|$?

- Γ_{j+1} small $\Rightarrow$ very few new Δ's created in $E_j \cup \Gamma_{j+1}$
- hence removal of few edges destroys all new Δ's

Finding Δ-free $\Gamma'_{j+1} \subseteq \Gamma_{j+1}$

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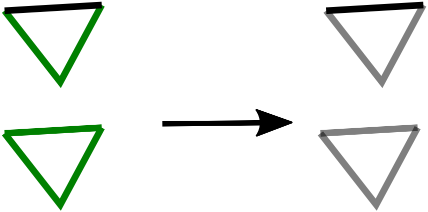
$E_j \cup \Gamma_{j+1}$ can create new Δ's:



Alteration to destroy new Δ's: $\Gamma'_{j+1} = \Gamma_{j+1} \setminus \mathcal{D}_{j+1}$

$\mathcal{D}_{j+1}$ = edges of a maximal edge-disjoint collection of bad pairs/triples

- easier to analyze than removing ≥ 1 edge from each new Δ
- $T_{j+1} = T_j \cup \Gamma'_{j+1}$ is Δ-free by maximality of $\mathcal{D}_{j+1}$

Open edges: effect of closed edges

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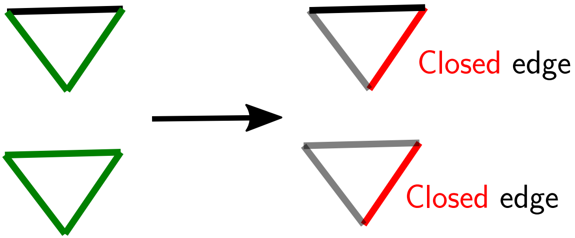
Updating "open edges" that can still be added

$O_{j+1} = O_j \setminus (\Gamma_{j+1} \cup \{\text{"closed edges"}\} \cup \{\text{extra edges for technical reasons}\})$

"Closed edge" forms a triangle with two edges in $E_{j+1} = E_j \cup \Gamma_{j+1}$:

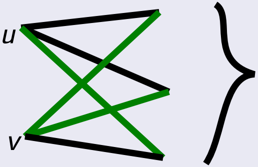


Open edges: self-stabilization mechanism

Updating "open edges" that can still be added

$O_{j+1} = O_j \setminus (\Gamma_{j+1} \cup \{\text{"closed edges"}\} \cup \{\text{extra random edges}\})$

$Y_e(j) = \#$ edges whose addition to Γ_{j+1} will close $e = \{u, v\}$



adding any of green edges closes $e = \{u, v\}$

Self-stabilization: make $\mathbb{P}(\text{closed})$ equal for all e (independent of history)

$\mathbb{P}(e \text{ not closed in next step of iteration}) \approx (1-p)^{|Y_e(j)|}$

$\mathbb{P}(e \text{ not (closed or extra edge)}) \approx (1-p)^{|Y_e(j)|} \cdot (1-q_e) \stackrel{!}{=} \text{same for all } e$

Summary: Semi-random construction of Δ-free subgraph

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Idea of each step (= iterated alteration approach)

(1) Generate few random edges $\Gamma_{j+1} \subseteq O_j$
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(3) Update $O_{j+1} \subseteq O_j \setminus \Gamma_{j+1}$

Number of edges between two large sets

Assume we can show

$$|O_j(A,B)| \approx q_j|A||B|, \text{ where } q_j = \Psi'(j\sigma), \text{ for } O_0 = H = K_n.$$

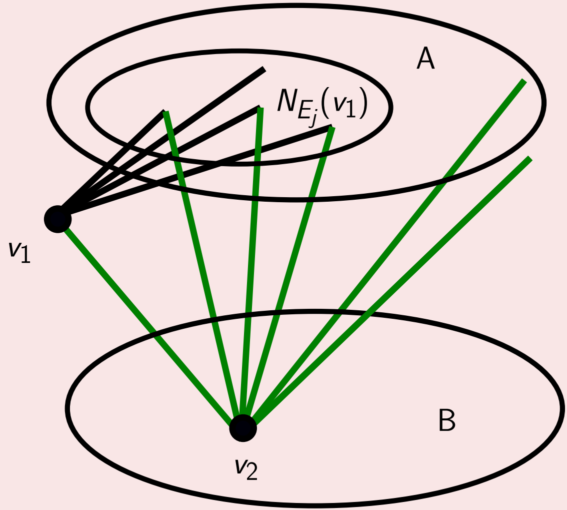
Use $p = \sigma/\sqrt{n}$, then we can approximate $|T_J(A,B)|$

$$\begin{aligned}
|T_J(A,B)| &= \sum_{0\le j<J} |T_{j+1}(A,B) \setminus T_j| \approx \sum_{0\le j<J} |\Gamma_{j+1}(A,B)| \\
&\approx \sum_{0\le j<J} p|O_j(A,B)| \approx \frac{1}{\sqrt{n}} \sum_{0\le j<J} \sigma q_j \cdot |A||B| \\
&\approx \frac{1}{\sqrt{n}} \int_0^{J\sigma} \Psi'(x)dx \cdot |A||B| \approx \frac{\Psi(J\sigma)}{\sqrt{n}}|A||B| \\
&\approx \frac{\sqrt{\beta(\log n)}}{\sqrt{n}}|A||B| = \varrho|A||B|
\end{aligned}$$

A technical difficulty

Difficulty of tracking $|O_j(A, B)|$

Choosing one edge into Γ_{j+1} may cause large change of $|O_j(A, B)|$:



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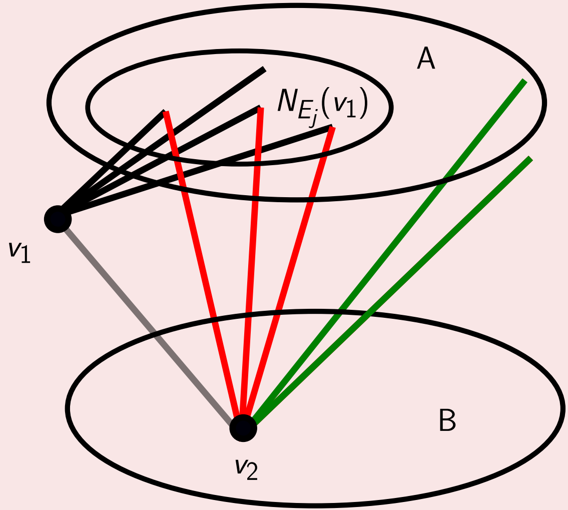

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Remarks

- Natural algorithmic packing version of Kim's $R(3,t)$ construction
- Establishes $s_r(K_3) = \Theta(r^2 \log r)$ asymptotics conjectured by Fox et.al.

Questions

- Further applications of the K_3-free packing result?
- Generalization of packing-result to K_k-free graphs worth effort?