Differential Equations (2403) Fall 2006
Exam 1

Name : SOLUTIONS

General Instructions:

For notes, you may only use the single–sided 8.5×11 sheet of notes that you have submitted by Monday, 25 September (if it was approved).

You may not use a calculator or any other electronic equipment.

You may not look at anyone else's paper or use any books or extra materials.

Use only the scrap paper at the back of the exam.

If you have any questions, raise your hand.

If you have time, CHECK ALL ANSWERS.

Part I: Skills

Solve THREE of the following. Put a big × through the number of the one you do not want graded. If you do not cross one off, I will grade the first three.

1. (2pt) Find the particular solution of:

$$y' = -4y, \qquad y(0) = 9.$$

$\boxed{y(x) = 9e^{-4x}}$

Check

$y' = 9(-4)e^{-4x} = -4(9e^{-4x})$

$= -4y$ ✓

$y(0) = 9$ ✓

2. (2pt) Find the particular solution of:

$$y' = x + y - 1, \qquad y(0) = 9.$$

$\boxed{y = 9e^{x} - x}$

$v = x + y \Rightarrow y' = v' - 1$

$\Rightarrow v' - 1 = v - 1$

$v' = v$

$v = ce^{x}$

$y = ce^{x} - x$

$9 = c$

$y = 9e^{x} - x$

Check

$y' = 9e^{x} - 1$

$= y + x - 1$ ✓

$y(0) = 9$ ✓

3. (2pt) Find an implicit formula for the general solution of:

$$(\cos y - y^2)dx - (x \sin y + 2xy)dy = 0.$$

$x \cos y - x y^2 = C_1$

Exact ?

$-\sin y - 2y = -\sin y - 2y$ ✓

$F(x,y) = x\cos y - xy^2 + g(y)$

$g(y) = C$

$\Rightarrow x\cos y - xy^2 = C_1$ is an implicit formula for sol.

4. (2pt) Find the particular solution of:

$$y'' - 25y = 0$$
$$y(0) = 2$$
$$y'(0) = -15.$$

$y = 2\cosh(5x) - 3\sinh(5x)$

Check : $y' = 10 \sinh 5x - 15 \cosh 5x$

$y'' = 50 \cosh 5x - 75 \sinh 5x$

$= 25(2\cosh 5x - 3\sinh 5x)$

$= 25y$ ✓

$y(0) = 2$ ✓

$y'(0) = -3(5) = -15$ ✓

Part II: Existence and Uniqueness
Answer ALL of the following.

1. (2pt) Consider the problem:

$$y' = f(x, y)$$
$$y(0) = 5.$$

Give sufficient conditions in order for there to exist a unique solution near the point

$$(x, y) = (0, 5).$$

It is sufficient if f and $\frac{\partial f}{\partial y}$ are continuous in an open neighborhood containing $(0,5)$.

2. (2pt) Consider the problem:

$$y'' + p(x)y' + q(x)y = 0$$
$$y(0) = 5$$
$$y'(0) = 2.$$

Give sufficient conditions for there to exist a unique solution near the point

$$(x, y) = (0, 5).$$

It is sufficient if p and q are continuous on an open interval containing $x = 0$.

3. We know from the theory of linear equations that the solution to the following initial value problem exists and is unique on the whole real line.

$$y' = y$$
$$y(0) = 2.$$

As we have discussed, the same is not true for nonlinear equations.

(a) (2pt) Determine the maximum interval of existence on which the solution to the following initial value problem exists and is unique. Show all work.

$$y' = y^2$$
$$y(0) = 4.$$

$(-\infty, 1/4)$

$\frac{dy}{y^2} = dx$

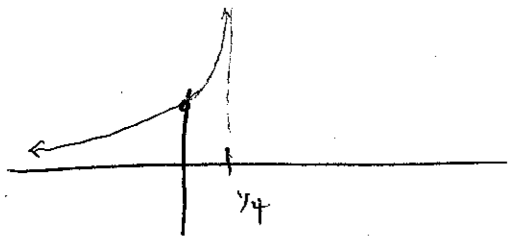
$-\frac{1}{y} = x + C$

$y = \frac{-1}{x+C}$

$4 = \frac{-1}{C}$

$C = -\frac{1}{4}$

$y = \frac{-1}{x - 1/4}$



(b) (2pt) Even when the existence and uniqueness theorem does not guarantee the existence of a unique solution, a solution may exist, and it may be unique. Show directly that a solution to the following problem **does exist**, but that it is **not unique**.

$$y' = \sqrt{y}$$
$$y(0) = 0.$$

The sol. exists b/c $y_1 \equiv 0$ is a sol. It is not unique. For instance, $y_2(x) = \begin{cases} 0 & x < 0 \\ x^2/4 & x \geq 0 \end{cases}$ is also a solution.

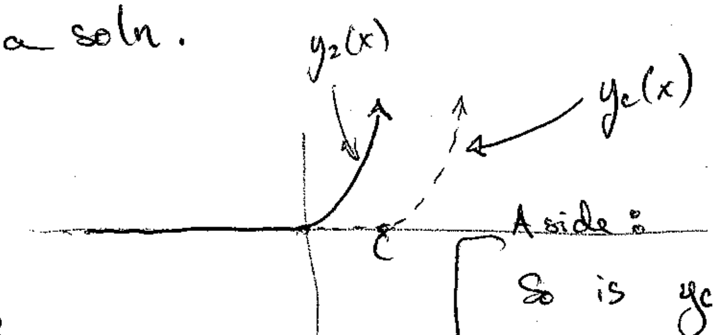
$y_1 \equiv 0$ is a soln.

$\frac{dy}{\sqrt{y}} = dx$

$2\sqrt{y} = x + C$

$y = \left(\frac{x+C}{2}\right)^2$

$y_2(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & x \geq 0 \end{cases}$ is a sol



Aside: So is $y_c(x)$, where

$y_c(x) = \begin{cases} 0 & x < C \\ \left(\frac{x-C}{2}\right)^2 & x \geq C \end{cases}$

Part III: Modelling

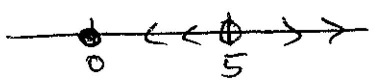
Answer ONE of the following questions. Put a big × through the number of the one you are not submitting for a grade. If you do not cross one off, I will grade the first one.

1. Suppose that the population of alligators in the Okefenokee swamp has a birth rate (i.e. births per unit population per unit time) $B(t) = 4P(t)$ and a death rate (i.e. deaths per unit population per unit time) $D(t) = 20$.

   (a) (2pt) Write down the ODE for $P(t)$.

   $$\dot{P} = 4P^2 - 20P$$

   (b) (2pt) Either by using a phase line or solving the problem directly, tell what happens to the population as $t \to \infty$ if $P(0) = 4$.

   $P(t) \to 0$ as $t \to \infty$.

   (c) (2pt) Either by using a phase line or solving the problem directly, tell what happens to the population as $t \to \infty$ if $P(0) = 10$.

   $P(t) \to \infty$ as $t \to \infty$.

   (d) (2pt) If $P(0) = 0$, then $P(t) = 0$ for all t. If $P(0) \neq 0$, can P ever be zero? i.e. Can there be a time $T^* < \infty$ such that $P(T^*) = 0$? Why or why not?

   No. If $P_1(t) \equiv 0$ and $P_2(t)$ is another solution, and

   $$P_1(T^*) = P_2(T^*) = 0,$$

   that would violate the existence & uniqueness theorem (which applies since $f(P) = 4P^2 - 20P$ and $f'(P) = 8P - 20$ are both continuous on $\mathbb{R}$.)

2. A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 3 gal/min, and the (perfectly mixed) solution leaves the tank at 4 gal/min.

   (a) (2pt) Write down an ODE model for the amount x of salt in the tank at time t.

   $$\dot{x} = 3 - \frac{4x}{V(t)} = 3 - \frac{4x}{60-t}$$

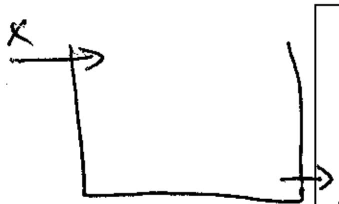
   (b) (2pt) When is the tank empty?

   $t \geq 60$ min

   (c) (2pt) How much salt is in the tank after t minutes?

   $$x(t) = 60 - t - \frac{1}{60^3}(60-t)^4$$

   (d) (2pt) Why is the problem is harder if we remove the assumption that the water in the tank is perfectly mixed?



   Then the salt concentration near the input and output are different. The concentration is a function of space as well as time. The problem would be a partial differential eqn (instead of an ODE).

$$\rho(t) = e^{4\int \frac{1}{60-t}\,dt} = e^{-4\ln(60-t)} = \frac{1}{(60-t)^4}$$

$$\frac{\dot{x}}{(60-t)^4} + \frac{4x}{(60-t)^5} = \frac{3}{(60-t)^4}$$

$$\frac{x}{(60-t)^4} = \int \frac{3}{(60-t)^4}\,dt = \frac{1}{(60-t)^3} + C$$

$$x = 60 - t + C(60-t)^4$$

$$x(0) = 0$$

$$0 = 60 + C\,60^4$$

$$C = -\frac{1}{60^3}$$

$$x = 60 - t - \frac{1}{(60)^3}(60-t)^4$$

Extra Credit (2pts, no credit given for sloppy or incomplete answers):
Consider the linear equation

$$y'' + P(x)y' + Q(x)y = 0.$$

Suppose that y_1 and y_2 are solutions. Find a first–order ODE satisfied by the Wronskian of y_1 and y_2. (Your final answer should have no y's in it.) Conclude that the Wronskian is either never zero or identically equal to zero.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$W' = y_1 y_2'' - y_2 y_1''$$

$$= y_1(-P y_2' - Q y_2) - y_2(-P y_1' - Q y_1)$$

$$= -P\, y_1 y_2' + P y_2 y_1'$$

$$= -P W$$

$$\boxed{W' = -P(x) W}$$

$$\frac{dW}{W} = -P(x)\,dx$$

$$\boxed{W = W_0 e^{-\int P(x)\,dx}}$$

The exp. is never zero

$\Rightarrow$ if $W(0) = W_0 = 0$, then $W(x) \equiv 0$

if $W_0 \neq 0$, then $W(x)$ is never zero.