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Bounds on vertical heat transport for infinite Prandtl number Rayleigh-Bénard convection

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For the infinite Prandtl number limit of the Boussinesq equations, the enhancement of vertical heat transport in Rayleigh-Bénard convection, the Nusselt number Nu, is bounded above in terms of the Rayleigh number Ra according to $Nu \leqslant .644 \times Ra^{\frac{1}{3}} [\log Ra]^{\frac{1}{3}}$ as $Ra \to \infty$. This result follows from the utilization of a novel logarithmic profile in the background method for producing bounds on bulk transport together with new estimates for the bi-Laplacian in a weighted L^2 space. It is a quantitative improvement over the best currently available analytic result, and it comes within the logarithmic factor of the pure 1/3 scaling anticipated by both the classical marginally stable boundary layer argument and the most recent high-resolution numerical computations of the optimal bound on Nu using the background method.

1. Introduction

Thermal convection processes play an important role in a wide range of phenomena in engineering, meteorology, oceanography, geophysics and astrophysics. Rayleigh-Bénard convection, where a fluid layer between rigid plates is heated from below and cooled from above, has emerged as a fundamental paradigm of nonlinear dynamics including instabilities and bifurcations, pattern formation, chaotic dynamics and fully developed turbulence (Kadanoff 2001). One of the key features of Rayleigh-Bénard convection is the enhancement of heat transport characterized by the Nusselt number Nu, the ratio of the total heat flux to the purely conductive heat flux in the absence of fluid flow. The dependence of Nu on the Rayleigh number Ra, the usual nondimensional measure of the applied temperature gradient, is of particular interest. For fluids presumed to be well-described by the Boussinesq approximation, the single nondimensional material parameter that distinguishes different substances is the Prandtl number Pr, the ratio of the fluid's kinematic viscosity to its thermal diffusivity. In this paper we derive mathematically rigorous and quantitatively precise estimates for Nu as a function of Ra for the Boussinesq model in the infinite Prandtl number limit.

The derivation of physically relevant upper bounds for convective heat transport has a long history beginning with Howard's seminal variational formulation (Howard 1963; Howard 1972) and Busse's subsequent multiple boundary layer theory (Busse 1969; Busse 1978). A different approach to the analysis that has come to be known as the "background method" was introduced several decades later (Doering & Constantin 1992; Doering & Constantin 1996). For Rayleigh-Bénard convection in arbitrary Prandtl number fluids, each method produces a high Rayleigh number bound on Nu proportional to $Ra^{\frac{1}{2}}$ uniformly in Pr. This 1/2-scaling (perhaps also with logarithmic modifications) has been conjectured as the ultimate high Rayleigh number behavior for fixed finite Pr (Kraichnan 1962; Spiegel 1971; Grossmann & Lohse 2000), albeit with different Pr dependence for the prefactor. Current experimental results for high-Ra are somewhat controversial (Chavanne et al. 1997; Glazier et al. 1999; Sommeria 1999; Niemela et al. 2000; Roche et al. 2001; Chilla, Rastello & Chaumat 2004).

A compelling marginally stable boundary layer argument predicts the "classical" scaling $Nu \sim Ra^{\frac{1}{3}}$ (Malkus 1954; Howard 1964) that is believed to persist into the asymptotically high-Ra regime for infinite Prandtl number fluids (Grossmann & Lohse 2000). Moreover, there are indications from recent direct numerical simulation and laboratory experiments that the 1/3 scaling applies over significant ranges of Ra even for $Pr = \mathcal{O}(1)$ (Amati et al. 2005; Nikolaenko et al. 2005; Funfschilling et al. 2005). The first upper bounds with 1/3 scaling exponent for infinite Pr Rayleigh-Bénard convection were proposed by Chan, who applied a sophisticated version of Busse's multiple boundary layer theory in the context of Howard's approach (Chan 1971). The utilization of additional assumptions in the asymptotic analysis, however, meant that Chan's prediction could not be regarded as a proof. Rigorous analysis using the background method produced an upper bound $\sim Ra^{\frac{2}{5}}$ (Doering & Constantin 2001) and the 2/5 exponent was subsequently shown to be sharp for the background method restricted to monotonic background temperature profiles (Otero 2002; Plasting 2004). The exponent was later lowered further to 4/11 using additional information from the maximum principle for the temperature equation (Yan 2004), but the best rigorous high-Ra bound on Nu previously proven was $\sim Ra^{\frac{1}{3}}[\ln Ra]^{\frac{2}{3}}$ employing both the maximum principle and a delicate singular-integral analysis (Constantin & Doering 1999).

Most recently an extremely careful and sophisticated numerical and boundary layer analysis of the variational problem for the optimal background profile has both clarified the subtlety of Chan's asymptotic theory and reaffirmed the pure scaling bound, predicting a high-Ra limit of the form $Nu \leqslant .139 \times Ra^{\frac{1}{3}}$ (Ierley, Kerswell & Plasting 2005). That study has confirmed that the optimal background profile is indeed *not* monotonic. In this paper we will use the conventional background analysis—no need for the maximum principle—and a nonconventional but nevertheless simple non-monotonic background profile with a "log-layer" in the bulk to derive the asymptotic $Ra \to \infty$ bound

$$Nu \leqslant .644 \times Ra^{\frac{1}{3}} [\ln Ra]^{\frac{1}{3}}. \tag{1.1}$$

This new analytical result relies on a novel estimate for the bi-Laplacian operator relating the vertical velocity to temperature fluctuations in a weighted L^2 function space. While (1.1) represents a modest quantitative improvement over the best previously available rigorous analytic (i.e., not relying on computational solutions or additional asymptotic hypotheses) bound, the utilization of a nonmonotonic profile to qualitatively improve the estimate is an important development for the background method.

The remainder of this paper is organized as follows. In the next section we set the stage by defining the model and the variables and reviewing the background method for bounding Nu in terms of Ra. In section 3 we construct the logarithmic background profile and apply the estimate for the bi-Laplacian operator to prove the stated bound (1.1). Section 4 contains a discussion of the result and some directions for further investigation. The key mathematical estimate necessary for the proof is derived in the Appendix.

2. Background method

The infinite Prandtl number limit of the Boussinesq equations for the temperature $T(x, y, z, t)$, the velocity $\boldsymbol{u}(x, y, z, t) = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ and the pressure $p(x, y, z, t)$ are

$$\dot{T} + \boldsymbol{u} \cdot \nabla T \;=\; \Delta T \tag{2.1}$$

$$\nabla p \;=\; \Delta \boldsymbol{u} + Ra\, \mathbf{k}\, T \tag{2.2}$$

$$0 \;=\; \nabla \cdot \boldsymbol{u}, \tag{2.3}$$

where Ra is the conventional Rayleigh number and we adopt the standard nondimensional variables. The spatial domain is $(x, y, z) \in [0, L_x] \times [0, L_y] \times [0, 1]$ with periodic boundary conditions in the horizontal (x and y) directions. In the vertical (z) direction the boundary conditions are $T = 1$ and $\boldsymbol{u} = 0$ at $z = 0$, and $T = 0$ and $\boldsymbol{u} = 0$ at $z = 1$. The no-slip velocity boundary conditions together with (2.3) imply $\partial_z w = 0$ for $z = 0$ and 1 as well. Recent research has rigorously established the quantitative validity of this model in the $Pr \to \infty$ limit of the full Boussinesq equations (Wang 2004).

Consider the decomposition

$$T(x, z, t) \;=\; \tau(z) + \theta(x, y, z, t), \tag{2.4}$$

where the "background" temperature profile $\tau(z)$ satisfies the inhomogeneous boundary conditions $\tau(0) = 1$ and $\tau(1) = 0$ so that the "fluctuation" $\theta(x, y, z, t)$ satisfies homogeneous boundary conditions at $z = 0$ and 1. Substituting (2.4) into (2.1) yields the fluctuation's evolution equation

$$\dot{\theta} + \boldsymbol{u} \cdot \nabla \theta \;=\; -w\,\tau' + \Delta\theta + \tau''. \tag{2.5}$$

The Nusselt number, the space-time averaged vertical heat flux in units of the steady state conduction heat flux, is

$$Nu \;:=\; \langle |\nabla T|^2 \rangle \;=\; \int_0^1 (\tau')^2 dz \;+\; \langle |\nabla\theta|^2 \rangle \;-\; 2\langle \theta \tau'' \rangle, \tag{2.6}$$

where $\langle \cdot \rangle$ denotes the space-time average defined by

$$\langle f \rangle \;=\; \limsup_{t \to \infty} \frac{1}{t} \int_0^t \frac{1}{L_x L_y} \int_0^{L_x} \int_0^{L_y} \int_0^1 f(x, y, z, t')\, dz\, dy\, dx\, dt'.$$

The last term on the right hand side of (2.6) was obtained via integration by parts in z. Averaging $2 \times \theta \times$ (2.5) with appropriate integrations by parts and adding (2.6) yields

$$Nu \;=\; \int_0^1 (\tau')^2\, dz - \langle |\nabla\theta|^2 + 2\tau'\, w\theta \rangle. \tag{2.7}$$

The background method is based on the following observation: if the background profile $\tau(z)$ is chosen such that

$$Q^{(\tau)}\{\theta\} \;:=\; \langle |\nabla\theta|^2 + 2\tau'\, w\theta \;\rangle \;\geqslant\; 0 \tag{2.8}$$

for every $\theta(x, y, z)$ satisfying homogeneous boundary conditions (with the associated function $w(x, y, z)$ defined by (2.2), (2.3), and homogeneous boundary conditions), then the Dirichlet integral of $\tau(z)$ is an upper bound for the Nusselt number, i.e.,

$$Nu \leqslant \int_0^1 (\tau')^2\, dz\;. \tag{2.9}$$

The objective of the background variational method is to construct profiles that minimize (2.9) subject to the constraint (2.8).

Eliminating the pressure in (2.2) leads to a direct relationship between $\theta(x,y,z)$ and $w(x,y,z)$:

$$\Delta^2 w \;=\; -Ra\,\Delta_H\theta, \tag{2.10}$$

where $\Delta_H = \partial_x^2 + \partial_y^2$. In terms of horizontally Fourier transformed variables $\hat\theta_{\boldsymbol k}(z)$ and $\hat w_{\boldsymbol k}(z)$, this relation is

$$(k^2 - \frac{d^2}{dz^2})^2\hat w \;=\; Ra\,k^2\hat\theta,$$

and the constraint (2.8) is fulfilled if for every wavenumber $\boldsymbol k$

$$\hat Q^{(\tau)}_{\boldsymbol k}\{\hat\theta_{\boldsymbol k}\} \;:=\; \int_0^1 \left[\left|\frac{d\hat\theta_{\boldsymbol k}}{dz}\right|^2 + k^2|\hat\theta_{\boldsymbol k}|^2 + \tau'(\hat\theta_{\boldsymbol k}\hat w^*_{\boldsymbol k} + \hat\theta^*_{\boldsymbol k}\hat w_{\boldsymbol k})\right] dz \;\geqslant\; 0.$$

For the remainder of this paper the analysis proceeds wavenumber by wavenumber so from this point onward we drop the hats and subscripts on $Q^{(\tau)}$, $\theta(z)$ and $w(z)$. Henceforth the goal is to produce a background profile $\tau(z)$ with the smallest possible Dirichlet integral satisfying $\tau(0)=1$ and $\tau(1)=0$, and such that for every real $k^2>0$,

$$Q^{(\tau)}\{\theta\} \;=\; \int_0^1 \big(|\theta'|^2 + k^2|\theta|^2 + 2\tau'\mathrm{Re}[\theta w^*]\big)\,dz \;\geqslant\; 0 \tag{2.11}$$

for every complex-valued function $\theta(z)$ satisfying $\theta(0)=0=\theta(1)$, where $w(z)$ solves

$$k^4 w - 2k^2 w'' + w'''' \;=\; Ra\,k^2\theta \tag{2.12}$$

with boundary conditions

$$w(0)=0=w(1), \quad w'(0)=0=w'(1). \tag{2.13}$$

The first two terms in $Q^{(\tau)}$ are manifestly positive while the last is indefinite.

The conventional intuition for applications of the background method has been to choose test profiles such that $\tau'(z)\approx 0$ in the bulk with the prescribed values for τ at $z=0$ and $z=1$ enforced via thin boundary layers. As the width of the boundary layers decreases, the quadratic form $Q^{(\tau)}$ is increasingly positive but the Dirichlet integral of $\tau(z)$ is also increasingly large. The challenge is to minimize the Dirichlet integral of $\tau(z)$ subject to the constraint that $Q^{(\tau)}$ is a non-negative quadratic form. In this paper we will utilize a class of logarithmic background profiles $\tau(z)$ with terms $\sim \ln z$ in the bulk so that $\tau'(z)$ will involve terms $\sim z^{-1}$. The key to the result is that while the "stable" stratification in $\tau(z)$ would appear to increase the Dirichlet integral, the price paid is offset by a significant contribution to $Q^{(\tau)}$'s positivity from the $z^{-1}\mathrm{Re}[\theta(z)w(z)^*]$ terms in the bulk.

3. Logarithmic background profile and the upper bound

We restrict attention to the high-Ra behavior and hence tailor the analysis to the asymptotic regime; quantitative improvements in the bounds are certainly possible for lower values of Ra. The heat transport bound derived in this section depends crucially on the following fact that is proved in the appendix: For $\theta(z)\in L^2[0,1]$ and $w(z)$ satisfying (2.12) and (2.13),

$$\mathrm{Re}\int_0^1 \frac{\theta w^*}{z}\,dz \;\geqslant\; \frac{2}{Ra}\int_0^1 \frac{|w|^2}{z^3}\,dz. \tag{3.1}$$

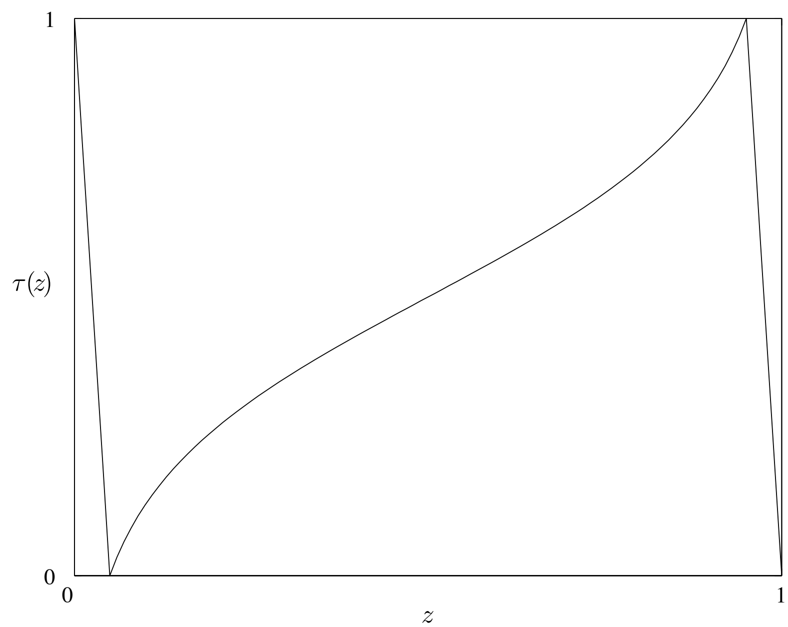

FIGURE 1. Background temperature profile used in the analysis. The boundary layers are linear in z while the profile $\tau(z) \sim \log(\frac{z}{1-z})$ in the bulk. The boundary layer thickness here is $\delta = \frac{1}{20}$.

Let $\delta \in (0, \frac{1}{2})$ and define the background profile

$$\tau(z) \;=\; \begin{cases} 1 - z/\delta, & 0 \leqslant z \leqslant \delta, \\ \frac{1}{2} + \lambda(\delta) \ln \frac{z}{1-z}, & \delta \leqslant z \leqslant 1-\delta, \\ (1-z)/\delta & 1-\delta \leqslant z \leqslant 1, \end{cases} \tag{3.2}$$

where

$$\lambda(\delta) \;=\; \frac{1}{2 \ln \frac{1-\delta}{\delta}}. \tag{3.3}$$

An example of this one-parameter family is illustrated in Figure 1. The boundary layer thickness δ will be small when Ra is large. The Dirichlet integral of $\tau(z)$ is easily evaluated:

$$\int_0^1 (\tau')^2 \, dz \;=\; \frac{2}{\delta} \times \left\{ 1 + \mathcal{O}\left(\frac{1}{[\ln \delta]^2} \right) \right\} \quad \text{as } \delta \to 0. \tag{3.4}$$

The objective now is to show that for a given (high) value of Ra, we may choose δ so that $Q^{(\tau)} \geqslant 0$ for all $\theta(z)$ with homogeneous boundary conditions. That value of δ may then be inserted into (3.4) to produce a bound.

The central idea is to use the stable stratification of $\tau(z)$ in the bulk to help dominate negative contributions to $Q^{(\tau)}$ from the boundary layers. Recalling (2.11),

$$Q^{(\tau)} = \underbrace{\int_0^{1/2} (|\theta'|^2 + k^2|\theta|^2)dz + 2\lambda \int_0^1 \frac{\mathrm{Re}[\theta w^*]}{z} dz - 2\int_0^\delta \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right) \mathrm{Re}[\theta w^*]\, dz}_{Q^{(\tau)}_{lower}}$$

$$+ \underbrace{\int_{1/2}^1 (|\theta'|^2 + k^2|\theta|^2)dz + 2\lambda \int_0^1 \frac{\mathrm{Re}[\theta w^*]}{1-z} dz - 2\int_{1-\delta}^1 \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right) \mathrm{Re}[\theta w^*]\, dz,}_{Q^{(\tau)}_{upper}}$$

where we have simply added and subtracted $\lambda/z + \lambda/(1-z)$ in the boundary layers, reorganized, and split $Q^{(\tau)}$ into two components. Identical analysis may be applied to each component independently; we carry it out explicitly only for the lower portion.

Dropping the $k^2|\theta|^2$ term and utilizing (3.1), we have

$$Q^{(\tau)}_{lower} \geqslant \int_0^{1/2} |\theta'|^2\, dz + \frac{4\lambda}{Ra}\int_0^1 \frac{|w|^2}{z^3}\, dz - 2\int_0^\delta \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right) |\theta|\, |w|\, dz. \quad (3.5)$$

The magnitude of the last term in (3.5) is restated and estimated

$$2\int_0^\delta \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right) |\theta|\, |w|\, dz = 2\int_0^\delta \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right) z^2 \left(\frac{|\theta|}{z^{1/2}}\right) \left(\frac{|w|}{z^{3/2}}\right) dz$$

$$\leqslant 2 \left(\sup_{0<z<1/2} \frac{|\theta(z))|}{z^{1/2}}\right) \left(\int_0^\delta z^4 \left[\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right]^2 dz\right)^{1/2} \left(\int_0^1 \frac{|w|^2}{z^3} dz\right)^{1/2}. \quad (3.6)$$

Because $\theta(0) = 0$, for $z \in [0, 1/2]$ we have

$$|\theta(z)| = \left|\int_0^z \theta'(\tilde{z})d\tilde{z}\right| \leqslant z^{1/2} \left(\int_0^{1/2} |\theta'(\tilde{z})|^2 d\tilde{z}\right)^{1/2},$$

so the supremum in (3.6) is bounded by the Dirichlet integral (over [0, 1/2]) of $\theta(z)$. Then applying Young's inequality to (3.6),

$$2\int_0^\delta \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right) |\theta|\, |w|\, dz$$

$$\leqslant \int_0^{1/2} |\theta'|^2 dz + \int_0^\delta z^4 \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right)^2 dz \times \int_0^1 \frac{|w|^2}{z^3}\, dz.$$

Inserting this into (3.5) we conclude that

$$Q^{(\tau)}_{lower} \geqslant \left[\frac{4\lambda}{Ra} - \int_0^\delta z^4 \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right)^2 dz\right] \times \int_0^1 \frac{|w|^2}{z^3}\, dz. \quad (3.7)$$

Noting that

$$\int_0^\delta z^4 \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right)^2 dz = \frac{\delta^3}{5} \times \left\{1 + \mathcal{O}\left(\frac{1}{|\ln\delta|}\right)\right\} \quad \text{as } \delta \to 0,$$

a sufficient asymptotic condition for the non-negativity of $Q^{(\tau)}_{lower}$ (and also $Q^{(\tau)}_{upper}$ and hence $Q^{(\tau)}$) is

$$Ra\, \delta^3 = 20\, \lambda = \frac{10}{\ln \frac{1-\delta}{\delta}}.$$

This is satisfied asymptotically by

$$\delta \sim \left(\frac{30}{Ra\ \ln Ra}\right)^{1/3} \quad \text{as } Ra \to \infty.$$

Inserting this into the Dirichlet integral for the upper bound we arrive at the result announced in (1.1):

$$Nu \leqslant \int_0^1 (\tau')^2\, dz \sim \frac{2}{\delta} \sim 2 \times \left(\frac{Ra\ \ln Ra}{30}\right)^{1/3} = .64366\cdots \times Ra^{1/3}\, [\ln Ra]^{1/3}.$$

□

4. Discussion

The upper bound on Nu derived in this paper is the best rigorous analytical heat transport estimate currently available for infinite Prandtl number convection. Modulo the $[\ln Ra]^{\frac{1}{3}}$ factor, it displays the classical $Ra^{\frac{1}{3}}$ scaling that was theoretically conjectured over half a century ago (Malkus 1954) and just within the last year shown via sophisticated computational and asymptotic analysis to be the best upper bound that can be expected from the background method (Ierley, Kerswell & Plasting 2005). The quantitative improvement of the bound derived here over previous results may be regarded as modest, and the presence of the logarithmic factor suggests that there is still room for progress, but nevertheless the analysis is important for understanding the mechanism by which the non-monotonic background profile improves the bound. The background method can be interpreted as a mathematically rigorous version of the marginally stable boundary layer theory, so the analysis result may be interpreted as a stability result for the background profile.

In its simplest form, the marginally stable boundary layer theory may be stated as follows: The boundary layer near the rigid walls where the fluid's motion is negligible is of a thickness δ such that as a convection layer in its own right, it is precisely marginally stable. Because the heat transport is totally conductive in the boundary layer, $Nu \sim \delta^{-1}$. Hence the goal is to deduce the Rayleigh number dependence of the boundary layer thickness. Presumably the boundary layer behaves like a Rayleigh-Bénard convection system with the wall temperature imposed on one side and the mean temperature imposed at the interface with the (turbulent) bulk, assumed to be approximately isothermal. While the velocity boundary conditions on the wall side of the boundary layers are definitely no-slip, it is not clear what the appropriate boundary condition at the bulk interface should be. In any case the stability criterion should be that the Rayleigh number based on the boundary layer parameters is equal to a critical Rayleigh number, an absolute constant for the purpose of this argument. Then because the boundary layer's Rayleigh number is proportional to δ^3 and the temperature drop across the boundary layer—half the total temperature drop—and all the other parameters are the same as for the full system characterized by the full Rayleigh number Ra, it is easy to conclude that $\delta \sim Ra^{-\frac{1}{3}}$.

In the background method on the other hand, the positivity condition (2.8) on the quadratic form $Q^{(\tau)}\{\theta\}$ is precisely the nonlinear energy stability condition (Joseph 1967; Straughan 1992)—at Rayleigh number $\frac{1}{2}Ra$—of the profile $\tau(z)$ if it was a steady conduction solution of the system. (Of course a heat source of the form $s(z) = -\tau''(z)$ would be required in the temperature equation for $\tau(z)$ to be a steady conduction solution.) Nonlinear energy stability is a sufficient condition for *absolute stability*, in contrast to linear theory which is capable of producing sufficient conditions for *instability*. These two stability criteria happen to coincide for basic Rayleigh-Bénard systems modeled by the Boussinesq equations with a linear conduction temperature profile, so the distinction does not enter into the heuristic considerations of the marginally stable boundary layer theory as described above.

In the language of stability theory, then, the background method employed in this work may be summarized by the statement, "If $\tau(z)$ is a nonlinearly stable steady temperature profile at Rayleigh number Ra, then its Dirichlet integral (proportional to the entropy production rate) is an upper limit to Nu at Rayleigh number $\frac{1}{2}Ra$." Thus the upper bound relies not just on the "stability of the boundary layer" — an ill-defined concept due to vaguely specified boundary conditions and the ambiguous notion of stability employed — but rather on the precisely defined nonlinear energy stability of the entire profile with the physical boundary conditions. The best upper bound is obtained by balancing this physical stability requirement with the minimal entropy production goal. In this light the emergence of the non-monotonic background profile may be understood in terms of stability enhancement. Boundary layers are clearly necessary at high Rayleigh numbers when the simple linear conduction profile is unstable. Stable stratification in the bulk inhibits convection there and provides a stable interface for the unstably stratified boundary layers, a much more stable interface than that produced by an isothermal core as in the naïve picture of the profile. The boundary layers must be kept sufficiently thin in order to ensure the stability of the entire profile, a requirement that naturally gives rise to the 1/3 scaling as in the heuristic argument. The quantitative price paid to ensure that the bulk provides a sufficiently robust interface to contain the boundary layers—at least in this rigorous analysis—is the additional logarithmic factor.

Non-monotonic mean temperature profiles with a stably stratified bulk have been observed in direct numerical simulations of turbulent convection, both for infinite Prandtl number (Sotin & Labrosse 1999) and for finite Prandtl numbers (Breuer et al. 2004). In the case of infinite Prandtl number we are able to enhance the nonlinear stability of background profiles with such a stably stratified bulk because of the strong positive correlation between the vertical velocity w and temperature fluctuation θ expressed in the explicit slaving of w to θ in (2.10) and (2.12). When the Prandtl number is finite, w is a dynamical variable in its own right so we do not have access to such tight control on the product of w and θ. Even so, the dynamics generates tight coupling of these variables. Indeed, the correlation of w and θ is precisely $Nu-1$, so the stronger the heat transport, the stronger the correlation. As the background method is currently formulated, though, we are unable to take advantage of this feature in conjunction with a non-monotonic profile to affect the scaling of the upper bounds. A thorough computational evaluation of the optimal background profile for the variational problem for arbitrary finite-Pr convection (Plasting & Kerswell 2003) yielded a monotonic profile and the $Ra^{\frac{1}{2}}$ scaling for the upper bound. Thus in order to exploit the increased stability that may be achieved with a stably stratified core to derive qualitatively improved bounds for finite Prandtl numbers, it is apparent that the background variational problem must be augmented with more information, i.e., constraints, derived from the full equations of motion to effectively express the enhanced of the coupling of w and θ.

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5. Appendix

Here we prove the key estimate (3.1) used to establish the positivity of $Q^{(\tau)}$ with the non-monotonic logarithmic background profiles. We begin the analysis by stating the

Proposition: If $0 < a < b < \infty$, the smooth function $w(z)$ satisfies

$$w(a) = 0 = w(b), \quad w'(a) = 0 = w'(b), \tag{5.1}$$

and $\theta(z)$ is defined by (2.12), then

$$\mathrm{Re} \int_a^b \frac{\theta w^*}{z}\, dz \;\geqslant\; \frac{2}{Ra} \int_a^b \frac{|w'|^2}{z}\, dz \;\geqslant\; \frac{2}{Ra} \int_a^b \frac{|w|^2}{z^3}\, dz. \tag{5.2}$$

These weighted inequalities are related to Muckenhoupt estimates for singular integrals (Stein 1993, Section 5.4). For the application in this work we need (3.1), stated here as a

Corollary: For $\theta(z) \in L^2[0,1]$ and $w(z)$ satisfying (2.12) and (2.13),

$$\mathrm{Re} \int_0^1 \frac{\theta w^*}{z}\, dz \;\geqslant\; \frac{2}{Ra} \int_0^1 \frac{|w|^2}{z^3}\, dz. \tag{5.3}$$

Proof of the Corollary: By a standard approximation argument we may assume that $\theta(z)$ and $w(z)$ are smooth. For $\epsilon > 0$ let $\theta_\epsilon(z)$ and $w_\epsilon(z)$ be $\theta(z-\epsilon)$ and $w(z-\epsilon)$ respectively. Then $w_\epsilon(z)$ satisfies (5.1) with $a = \epsilon$, $b = 1+\epsilon$, so the proposition is

$$\mathrm{Re} \int_\epsilon^{1+\epsilon} \frac{\theta_\epsilon w_\epsilon^*}{z}\, dz \;\geqslant\; \frac{2}{Ra} \int_\epsilon^{1+\epsilon} \frac{|w_\epsilon|^2}{z^3}\, dz.$$

Changing variables,

$$\mathrm{Re} \int_0^1 \frac{\theta w^*}{z+\epsilon}\, dz \;\geqslant\; \frac{2}{Ra} \int_0^1 \frac{|w|^2}{(z+\epsilon)^3}\, dz. \tag{5.4}$$

For $\theta(z) \in L^2$, the vertical velocity $w(z)$ satisfies $|w(z)| \leqslant Cz^2$ where C depends only on Ra, k^2 and the L^2 norm of $\theta(z)$. This follows from regularity theory for (2.12) plus Sobolev embedding; see Doering & Constantin (2001) for an explicit proof in this context. Hence $|\theta(z)w(z)^*|/z$ and $|w(z)|^2/z^3$ are integrable functions on $[0,1]$ so we may let $\epsilon \to 0$ to recover (3.1).

□

We will use two lemmas in the proof of the proposition. The first is

Lemma 1: For smooth functions $w(z)$ satisfying (5.1),

$$(i) \quad -\mathrm{Re} \int_a^b \frac{w'' w^*}{z} dz \;=\; \int_a^b z \left| \left(\frac{w}{z}\right)' \right|^2 dz,$$

$$(ii) \quad \mathrm{Re} \int_a^b \frac{w'''' w}{z} dz \;=\; \int_a^b z \left| \left(\frac{w}{z}\right)'' \right|^2 dz.$$

Proof of Lemma 1: Let $\zeta(z) := w(z)/z$ and note that $\zeta(z)$ satisfies the same homogeneous boundary conditions (2.13) as $w(z)$. Successive integrations (many by parts) yield

$$
\begin{aligned}
-\mathrm{Re}\int_a^b w''\,\frac{w^*}{z}\,dz \;&=\; \mathrm{Re}\int_a^b w'\,\left(\frac{w^*}{z}\right)'\,dz = \;\mathrm{Re}\int_a^b (z\,\zeta)'\,(\zeta^*)'\,dz \\
&= \int_a^b \left\{z\,|\zeta'|^2 \;+\; \mathrm{Re}[\zeta^*\,\zeta']\right\}\,dz \\
&= \int_a^b \left\{z\,|\zeta'|^2 \;+\; \frac{1}{2}\left(|\zeta|^2\right)'\right\}\,dz \;=\; \int_a^b z\,|\zeta'|^2\,dz.
\end{aligned}
$$

The proof of (ii) is similar:

$$
\begin{aligned}
\mathrm{Re}\int_a^b w''''\,\frac{w^*}{z}\,dz \;&=\; \mathrm{Re}\int_a^b w''\,\left(\frac{w^*}{z}\right)''\,dz = \;\mathrm{Re}\int_a^b (z\,\zeta)''\,(\zeta^*)''\,dz \\
&= \int_a^b \left\{z\,|\zeta''|^2 \;+\; 2\mathrm{Re}[\zeta'\,(\zeta^*)'']\right\}\,dz \\
&= \int_a^b \left\{z\,|\zeta''|^2 \;+\; \left(|\zeta'|^2\right)'\right\}\,dz \;=\; \int_a^b z\,|\zeta''|^2\,dz.
\end{aligned}
$$

□

The second lemma is

Lemma 2: Let $\zeta(z) \in C^\infty[a,b]$ satisfy the same homogeneous boundary conditions as $w(z)$ in (5.1). Then:

$$
\int_a^b \frac{|\zeta|^2}{z}\,dz \;\leqslant\; \left(\int_a^b z\,|\zeta|^2\,dz \;\times\; \int_a^b z\,|\zeta''|^2\,dz\right)^{1/2}. \tag{5.5}
$$

Proof of Lemma 2: Define $\phi(z) := \zeta(z)/z$ and observe that $\phi(z)$ satisfies the same homogeneous boundary conditions as $\zeta(z)$. Then the statement to be proved is

$$
\int_a^b z|\phi|^2\,dz \;\leqslant\; \left(\int_a^b z^3|\phi|^2 dz \times \int_a^b z\,|z\phi'' + 2\phi'|^2\,dz\right)^{1/2}. \tag{5.6}
$$

Notice first that

$$
\begin{aligned}
&\int_a^b z\,|z\phi'' + 2\phi'|^2\;dz \\
&= \int_a^b z^3|\phi''|^2\,dz \;+\int_a^b \left[\,2z^2\phi'(\phi^*)'' + 2z^2(\phi^*)'\phi'' \;\;+\;\; 4z|\phi'|^2\,\right] dz \\
&= \int_a^b z^3|\phi''|^2\,dz + \int_a^b \left(2z^2|\phi'|^2\right)'\,dz \;=\; \int_a^b z^3\,|\phi''|^2\,dz,
\end{aligned}
$$

so (5.6), to be proved, becomes

$$
\int_a^b z\,|\phi|^2\,dz \;\leqslant\; \left(\int_a^b z^3\,|\phi|^2\,dz \;\times\; \int_a^b z^3|\phi''|^2\,dz\right)^{1/2}. \tag{5.7}
$$

To see (5.7) it is sufficient to establish

$$\int_a^b z\,|\phi|^2\,dz \leqslant \left(\int_a^b z^3\,|\phi|^2\,dz \;\times\; \int_a^b z\,|\phi'|^2\,dz\right)^{1/2} \tag{5.8}$$

$$\text{and}\quad \int_a^b z\,|\phi'|^2\,dz \leqslant \int_a^b z^3\,|\phi''|^2\,dz. \tag{5.9}$$

For (5.8), integrate by parts and use the Cauchy-Schwarz inequality:

$$\begin{aligned}\int_a^b z\,|\phi|^2\,dz \;&=\; \int_a^b (z^2)'\,\frac{1}{2}|\phi|^2\,dz = \;-\mathrm{Re}\int_a^b z^2\,\phi^*\,\phi'\,dz\\ &\leqslant\; \left(\int_a^b z^3\,|\phi|^2\,dz \;\times\; \int_a^b z\,|\phi'|^2\,dz\right)^{1/2}.\end{aligned}$$

Likewise for (5.9),

$$\begin{aligned}\int_a^b z\,|\phi'|^2\,dz \;&=\; \int_a^b (z^2)'\,\frac{1}{2}(\phi')^2\,dz = \;-\mathrm{Re}\int_a^b z^2\,(\phi^*)'\,\phi''\,dz\\ &\leqslant\; \left(\int_a^b z\,|\phi'|^2\,dz \;\times\; \int_a^b z^3\,|\phi''|^2\,dz\right)^{1/2}.\end{aligned}$$

□

Proof of the Proposition: Using (2.12) and letting $\zeta(z) := w(z)/z$,

$$\begin{aligned}\mathrm{Re}\int_a^b \frac{\theta w^*}{z}\,dz \;&= \mathrm{Re}\int_a^b \frac{(k^4 w - 2k^2 w'' + w'''')\,w^*}{Ra\,k^2\,z}\,dz\\ &= \frac{k^2}{Ra}\int_a^b \frac{|w|^2}{z}\,dz - \frac{2}{Ra}\mathrm{Re}\int_a^b \frac{w'' w^*}{z}\,dz \;+\; \frac{1}{Ra\,k^2}\mathrm{Re}\int_a^b \frac{w'''' w^*}{z}\,dz\\ &= \frac{k^2}{Ra}\int_a^b z\,|\zeta|^2\,dz + \frac{2}{Ra}\int_a^b z\,|\zeta'|^2\,dz \;+\; \frac{1}{Ra\,k^2}\int_a^b z\,|\zeta''|^2\,dz\\ &\geqslant \frac{2}{Ra}\int_a^b z\,|\zeta'|^2\,dz + \frac{2}{Ra}\left(\int_a^b z\,|\zeta|^2\,dz \;\times\; \int_a^b z\,|\zeta''|^2\,dz\right)^{1/2},\end{aligned}$$

where we used Lemma 1 in the third line and Young's inequality (i.e., $a^2 + b^2 \geqslant 2ab$) in the fourth. Because $\zeta(z)$ satisfies the same boundary conditions as $w(z)$, Lemma 2 yields

$$\mathrm{Re}\int_a^b \frac{\theta\,w^*}{z}\,dz \;\geqslant\; \frac{2}{Ra}\int_a^b z\,|\zeta'|^2\,dz \;+\; \frac{2}{Ra}\int_a^b \frac{|\zeta|^2}{z}\,dz. \tag{5.10}$$

On the other hand $w(z) = z\zeta(z)$ implies

$$\int_a^b \frac{|w'|^2}{z}\,dz \;=\; \int_a^b \left[\frac{|\zeta|^2}{z} + 2\,\mathrm{Re}\,\zeta^*\zeta' + z\,|\zeta'|^2\right]\,dz = \int_a^b \frac{|\zeta|^2}{z}\,dz + \int_a^b z\,|\zeta'|^2\,dz, \tag{5.11}$$

so that combining (5.10) and (5.11) we deduce

$$\mathrm{Re}\int_a^b \frac{\theta\,w^*}{z}\,dz \;\geqslant\; \frac{2}{Ra}\int_a^b \frac{|w'|^2}{z}\,dz.$$

Finally,

$$\int_a^b \frac{|w|^2}{z^3}\,dz \;=\; -\int_a^b \left(z^{-2}\right)' \,\frac{1}{2}\,|w|^2\,dz = \;\mathrm{Re}\int_a^b \frac{w^* w'}{z^2}\,dz$$
$$\leqslant \left(\int_a^b \frac{|w|^2}{z^3}\,dz \;\times\; \int_a^b \frac{|w'|^2}{z}\,dz\right)^{1/2}$$

means that

$$\int_a^b \frac{|w|^2}{z^3}\,dz \;\leqslant\; \int_a^b \frac{|w'|^2}{z}\,dz.$$

□

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