

Slow Motion of Gradient Flows

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Abstract

We present sufficient conditions on an energy landscape in order for the associated gradient flow to exhibit slow motion or “dynamic metastability.” The first condition is a weak form of convexity transverse to the so-called slow manifold, \mathcal{N} . The second condition is that the energy restricted to \mathcal{N} is Lipschitz with a constant $\delta \ll 1$. One feature of the abstract result that makes it of broader interest is that it does not rely on maximum principles.

As an application, we give a new proof of the exponentially slow motion of transition layers in the one-dimensional Allen-Cahn equation. The analysis is more nonlinear than previous work: It relies on the nonlinear convexity condition or “energy-energy-dissipation inequality.” (Although we do use the maximum principle for convenience in the application, we believe it may be removed with additional work.) Our result demonstrates that a broad class of initial data relaxes with an exponential rate into a δ -neighborhood of the slow manifold, where it is then trapped for an exponentially long time.

Keywords. Energy methods, nonlinear partial differential equations, dynamic metastability, coarsening rates.

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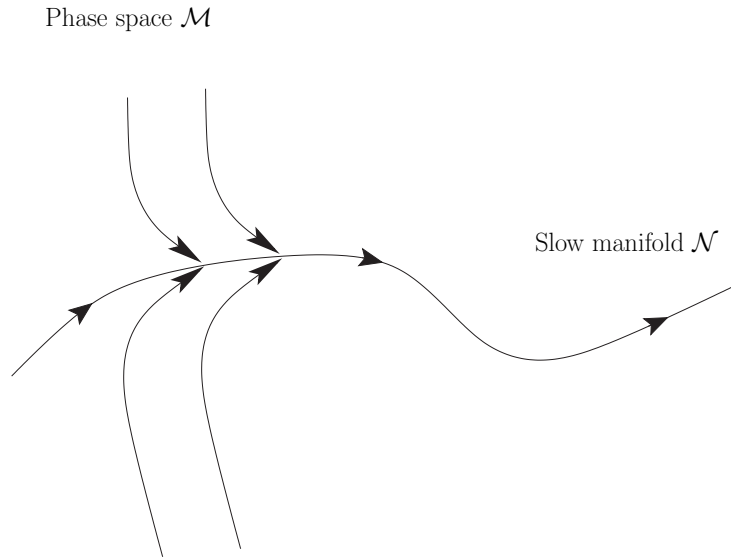


Figure 1: Fast relaxation to the slow manifold, slow motion along it.

1 Introduction

Although local energy minimizers are the only stable states of a gradient flow system, “dynamic metastability” is characterized by evolution so slow that solutions *appear* to be stable. Often this metastable behavior is misleading in the sense that after a long time, the solution undergoes drastic change.

Our goal is to convert information about the static structure of the energy landscape into information about the dynamics of the related gradient flow. The main result is a pair of sufficient conditions for metastable behavior (see Theorem 1.1 and Remark 1 below). We have in mind a PDE with gradient-flow structure. Viewing state space as an abstract manifold \mathcal{M} , metastability means that generic initial data is drawn quickly to a *slow manifold* $\mathcal{N} \subset \mathcal{M}$, where it remains trapped for a long time (Figure 1).

Abstract result: sufficient conditions for metastability. Consider the gradient flow

$$\begin{aligned} u_t &= -\nabla E(u) & t > 0, \\ u &= u(0) & t = 0. \end{aligned} \tag{1.1}$$

Our assumptions about the energy landscape are:

(i) For every $u \in \mathcal{M}$, there exists a $v \in \mathcal{N}$ such that

$$\frac{1}{2}|u - v|^2 \leq E(u) - E(v) \leq \frac{1}{2}|\nabla E(u)|^2. \quad (1.2)$$

(ii) There exists a finite constant δ such that for every $v_1, v_2 \in \mathcal{N}$,

$$|E(v_1) - E(v_2)| \leq \delta|v_1 - v_2|. \quad (1.3)$$

Assumption (i) is a weak form of strict convexity of E transverse to \mathcal{N} . Assumption (ii) is a Lipschitz condition on E restricted to \mathcal{N} . Our result is:

Theorem 1.1. *Suppose that Assumptions (i) and (ii) hold, and let v be such that $v(t)$ and $u(t)$ satisfy (1.2). Then the solution of (1.1) is drawn into a δ -neighborhood of \mathcal{N} with an exponential rate close to 1; that is, for any $\varepsilon \in (0, 1)$, there exists a finite constant C_ε such that*

$$\begin{aligned} & |u(t) - v(t)| + (E(u(t)) - E(v(t)))^{1/2} \\ & \lesssim \exp(-(1 - \varepsilon)t)(E(u(0)) - E(v(0)))^{1/2} + C_\varepsilon\delta. \end{aligned} \quad (1.4)$$

Moreover, we have for any $0 < s < t$ that

$$|u(t) - u(s)| \lesssim (E(u(s)) - E(v(s)))^{1/2} + \delta(t - s + 1). \quad (1.5)$$

Remark 1 (Metastability). *While there is no restriction on the constant δ in (1.3), it is in the case $\delta \ll 1$ that Theorem 1.1 reflects “dynamic metastability.” For initial data with an order one energy gap, the right-hand side of (1.4) is initially dominated by the first term. Setting $\varepsilon = 1/2$ in (1.4) and defining*

$$t_1 := \log \left(\frac{E(u(0)) - E(v(0))}{\delta^2} \right) \quad (1.6)$$

gives a timescale for the “initial layer” during which the energy gap is reduced from order one to

$$(E(u(t_1)) - E(v(t_1)))^{1/2} \stackrel{(1.4),(1.6)}{\lesssim} \delta. \quad (1.7)$$

After the initial layer comes the “slow motion phase” which lasts for a time of order δ^{-1} : Setting $s = t_1$ in (1.5), we have:

$$\begin{aligned} |u(t) - u(t_1)| & \lesssim (E(u(t_1)) - E(v(t_1)))^{1/2} + \delta(t - t_1 + 1) \\ & \stackrel{(1.7)}{\lesssim} \delta + \delta(t - t_1). \end{aligned}$$

That is, the change in u is of order δ until $(t - t_1) \sim \delta^{-1}$.

Remark 2 (Notation). We make occasional use of the symbols \ll and \gg to denote “much greater than” and “much less than.” For instance, Remark 1 says that if the Lipschitz constant is much less than one ($\delta \ll 1$), then the slow motion timescale is much greater than one ($\delta^{-1} \gg 1$).

We write $A(u) \lesssim B(u)$ if and only if there exists a finite, positive constant C such that

$$A(u) \leq C B(u).$$

(Analogous statements hold for \gtrsim and \sim .) In the abstract result (Theorem 1.1 and Section 2), the constant C is universal. In the application (Theorem 1.2 and Section 3), the constant depends at most on the potential G .

The proof of Theorem 1.1 is elementary and relies on the general inequality

$$\begin{aligned} |u(t) - u(t_0)|^2 &\leq \left(\int_{t_0}^t \left(-\frac{d}{ds} E(u(s)) \right)^{1/2} ds \right)^2 \\ &= \left(\int_{t_0}^t \frac{1}{w(s)^{1/2}} w(s)^{1/2} \left(-\frac{d}{ds} E(u(s)) \right)^{1/2} ds \right)^2 \\ &\leq \int_{t_0}^t \frac{1}{w(s)} ds \left(w(t_0) E(u(t_0)) - w(t) E(u(t)) \right. \\ &\quad \left. + \int_{t_0}^t \dot{w}(s) E(u(s)) ds \right), \end{aligned} \tag{1.8}$$

for any positive weight w and any $t \geq t_0$. For (1.4), we use $w \equiv 1$ and develop a differential inequality for the energy gap between u and v . For (1.5), we use (1.4) and equation (1.8) with two different weights: an exponential weight for the “initial layer” of rapid energy relaxation, and a constant weight for the stagnant phase.

Application to Allen-Cahn: background. A classic example of metastable behavior is the exponentially slow motion of transition layers in the one-dimensional Allen-Cahn equation,

$$u_t = u_{xx} - G'(u) \quad \text{for } x \in (0, L), t > 0. \tag{1.9}$$

We will show how the abstract result of Theorem 1.1 may be applied to give a new proof of the exponentially slow coarsening time-scale. Moreover, the result (Theorem 1.2, below) shows that closeness to the slow manifold is not only *propagated*, but also *generated*: A broad class of initial data is quickly

drawn into a small neighborhood of the slow manifold, where it is then trapped for an exponentially long time. We now give some brief background; for a more thorough introduction, see for instance [CP, FH, BK, W, C].

The Allen-Cahn equation (1.9) is the L^2 -gradient flow for the scalar Ginzburg-Landau energy,

$$E(u) = \int_0^L \left(\frac{1}{2} u_x^2 + G(u) \right) dx. \quad (1.10)$$

The potential, G , has nondegenerate minima at two preferred phases, which for simplicity we normalize as $u = \pm 1$. To be precise, we assume that G is a smooth, even potential satisfying

$$\begin{aligned} G(u) &> 0 & u \neq \pm 1 & & \text{and} & & G(\pm 1) &= 0, \\ G'(u) &\leq 0 & u \in [0, 1] & & \text{and} & & G''(\pm 1) &> 0. \end{aligned} \quad (1.11)$$

A standard choice of potential is

$$G(u) := \frac{(1 - u^2)^2}{4}. \quad (1.12)$$

The two characteristic length-scales of equation (1.9) are the minimal distance ℓ between zeros of u and the width of an optimal transition layer between the preferred phases, which is order one in our scaling of the equation. In the case in which there is a clear separation between the two scales – i.e. $\ell \gg 1$ – states with bounded energy are characterized by large regions of $u \approx \pm 1$, separated by order one interfaces on which the energy concentrates.

It is the degeneracy of the energy for large ℓ that makes the motion slow. By degeneracy, we mean that for a finite-sized system with well-separated transition layers, the change in energy from translating an interface is exponentially small with respect to the distance between layers. Thus, until two layers come close, not much energy is dissipated, and if there is not much energy dissipated, then the interfaces hardly move. Driven only by the exponentially small correction terms to the energy, the motion is exponentially slow.

Thus, the heuristics suggest three distinct stages for the evolution problem: A fast, initial stage of energy relaxation, an exponentially slow stage of layer motion, and a collision stage in which the two closest layers come together and annihilate. Then the process repeats. A detailed analysis of the exponentially slow motion of transition layers was carried out by Carr and Pego [CP] and Fusco and Hale [FH]. Subsequently, Ward [W] studied all three stages using a combination of numerical and asymptotic methods. Eckmann and Rougemont [ER] and Rougemont [R] studied the coarsening problem on \mathbb{R} , analyzing also

the collision stage. Most recently, Chen [C] presented a result that also includes the initial relaxation stage: He proved that initial data that is order one away from the slow manifold is drawn into a small neighborhood of it and then trapped in the slow motion phase. The proof uses an idea of de Mottoni and Schatzman [MS] and a result of Fife and McLeod [FM] on the stability of the travelling wave solution on \mathbb{R} .

Bronsard and Kohn [BK] introduced an alternate, energy-based analysis: Via an elementary method requiring weaker hypotheses than [CP] but returning weaker results, they prove that initial data that is algebraically close in energy to the slow manifold stays close for an algebraically long time. In an extension, Grant [G] proved that initial data that is exponentially close to the slow manifold stays close for an exponentially long time.

Here, we use an energy-based method – natural for a gradient flow – to derive stronger information: Namely, we start with initial data whose energy is order one away from the slow manifold and capture the fast, initial relaxation, followed by the exponentially long stage of layer motion. Thus, the main result is similar to Chen’s, but the method is different. Perhaps the most salient feature of our method is that it exploits the nonlinearity: By passing from the linearized estimates of energy and energy dissipation (which appear already in [CP]) to their nonlinear counterparts, we gain a strong advantage; see the discussion just after Theorem 1.2 below for a heuristic illustration.

Remark 3. *For convenience, we rely on the maximum principle in two ways. First, a maximum of the initial data that is greater than 1 (resp. minimum less than -1) is driven exponentially quickly to 1 (resp. -1); for simplicity, we assume throughout the paper that $u \in [-1, 1]$. Second, the zeros of (1.9) move continuously and can only decrease in number [A]; we use this fact when proving energy-energy-dissipation. We emphasize that the abstract result is independent of the maximum principle, so that Theorem 1.1 may be used for higher order equations or systems, where the maximum principle does not hold.*

Application to Allen-Cahn: result. We state our result for the Allen-Cahn equation for the case of initial data with two zeros and periodic boundary conditions. This is for simplicity; one can generalize to N zeros, in which case the timescale for motion is controlled by the distance between the two nearest zeros. The “energy-optimal profiles” play the role of the slow manifold:

Definition 1. *We call v an energy-optimal profile with zeros x_1, \dots, x_N ($N \in \mathbb{N}$) if v has simple zeros at x_1, \dots, x_N and minimizes the energy (1.10) in-between zeros.*

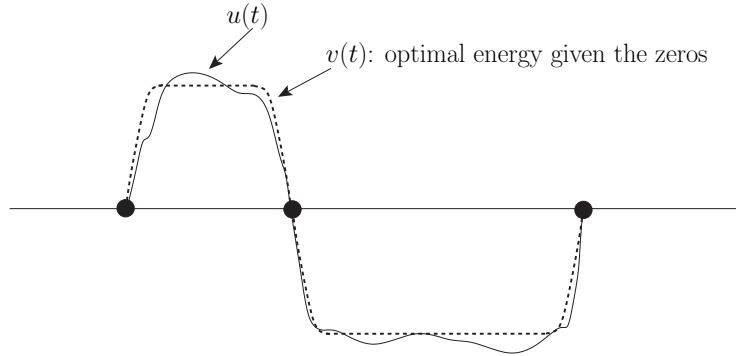


Figure 2: We associate to u the function v that has the same zeros, the same sign in-between zeros, and minimal energy.

Associated to u is the periodic energy-optimal profile v that has the same zeros as u and the same sign as u in-between zeroes (Figure 2). We remark that v is continuous but typically not differentiable.

We assume that the energy of the initial data is bounded by $4c_0$, where c_0 is the energy of an optimal transition layer on \mathbb{R} ,

$$c_0 := \sqrt{2} \int_{-1}^1 G(u)^{1/2} du.$$

This makes sense since we have in mind the successive collision and annihilation of neighboring layers, and just after a collision event has reduced the number of layers from four to two, the energy is close to $4c_0$. This assumption means that our constants are universal, depending only on G . (One can instead allow for any order one energy E_0 of the initial condition, and then the constants depend also on E_0 .) We formulate our result as:

Theorem 1.2. *Let G satisfy (1.11). There exist positive constants ℓ_* and ϵ with the following property. Let $u : [0, L] \times (0, \infty) \rightarrow \mathbb{R}$ be a smooth, periodic solution of*

$$\begin{aligned} u_t &= u_{xx} - G'(u) & x \in (0, L], t > 0, \\ u &= u_0 & x \in (0, L], t = 0. \end{aligned} \tag{1.13}$$

Suppose that u_0 has exactly two simple zeros, $x(0)$, $y(0)$, and that

$$|u_0(x)| \leq 1, \quad E(u_0) \leq 4c_0, \quad \ell(0) \geq \ell_*, \tag{1.14}$$

where $\ell(t)$ denotes the minimal distance between the zeros of $u(t)$ (bearing in mind the periodicity). Then for all

$$t \leq \epsilon \exp(\sqrt{G''(1)}\ell(0)),$$

$u(t)$ has exactly two simple zeros, $x(t)$, $y(t)$, and

$$|x(t) - x(0)| + |y(t) - y(0)| \lesssim 1. \quad (1.15)$$

Moreover, not only are the zeros of u trapped within an order one neighborhood of their starting location, but after an initial layer, u is trapped exponentially close in H^1 to the corresponding energy-optimal profile: For all t with

$$\epsilon^{-1}\ell(0) \leq t \leq \epsilon \exp(\sqrt{G''(1)}\ell(0)),$$

we have

$$\|u(t) - v(t)\|_{H^1([0,L])} \lesssim \exp(-\sqrt{G''(1)}\ell(0)). \quad (1.16)$$

The main ingredient in Theorem 1.2 is energy-energy-dissipation. Roughly, the idea is: Suppose u is a solution of (1.13) where the initial data $u(0)$ has N zeros. Suppose that the optimal energy of a function with N zeros on $(0, L)$ is well-approximated by $c_0 N$. By the gradient flow structure, we have

$$\frac{d}{dt}(E(u(t)) - c_0 N) = - \int_0^L (u_{xx} - G'(u))^2 dx. \quad (1.17)$$

Now suppose that we have the energy-energy-dissipation inequality:

$$\int_0^L (u_{xx} - G'(u))^2 dx \geq \frac{1}{C}(E(u(t)) - N c_0).$$

Then (1.17) becomes

$$\frac{d}{dt}(E(u(t)) - c_0 N) \leq -\frac{1}{C}(E(u(t)) - N c_0).$$

This estimate implies the relaxation of the energy to $c_0 N$ with an exponential rate in time. After this relaxation, u is very close in energy to the optimal N -layer configuration. Subsequently, because small translations of the zeros hardly change the energy, one expects u to be trapped for a long time, its zeros barely moving.

To make these ideas precise, we will apply the abstract result of Theorem 1.1. Therefore, the first step is to establish Assumptions (i) and (ii). Energy-energy-dissipation takes the form:

Proposition 1.1. *Let G satisfy (1.11). There exist constants $\ell_1, C_1 < \infty$ with the following property. Suppose $u : [0, L] \rightarrow \mathbb{R}$ is a periodic function that satisfies the conditions:*

$$\text{The minimal distance between simple zeros of } u \text{ is at least } \ell_1. \quad (1.18)$$

$$\text{The energy between adjacent zeros is less than } (G(0)/2)\ell_1. \quad (1.19)$$

Then u and the corresponding energy-optimal profile v satisfy

$$\frac{1}{2C_1} \|u - v\|_{H^1([0,L])}^2 \leq E(u) - E(v) \leq \frac{C_1}{2} \int_0^L (u_{xx} - G'(u))^2 dx. \quad (1.20)$$

Proposition 1.1 says that u and the energy-optimal profile satisfy a relationship of the form (1.2) (i.e. they satisfy (1.2) and (1.1) after rescaling the energy according to $\tilde{E} := C_1 E$ and time according to $\tilde{t} = t/C_1$). The hypothesis (1.18) of well-separated zeros allows us to compare v to its infinite-system limit (cf. Subsection 3.2). The hypothesis (1.19) allows us to bound u away from $u \equiv 0$ (cf. Lemma 3.9).

Verification of the Lipschitz condition (1.3) follows from direct calculations on the energy. The Lipschitz property, stated as Lemma 3.1, was proved already in [CP], Section 7, but for completeness, we include a proof in Subsection 3.1.

Organization. The proof of Theorem 1.1 is given in Section 2. In Section 3 we apply the abstract theorem to the Allen-Cahn equation. The proof of Theorem 1.2 is given in Subsection 3.1, assuming energy-energy-dissipation (Proposition 1.1) and the Lipschitz condition on the slow manifold (Lemma 3.1). Then to prove the energy-energy-dissipation relationship, we begin by proving the linearized estimates in Subsection 3.2. In Subsection 3.3, we show how to extend from the linear to the nonlinear estimates, proving Proposition 1.1. Finally, in Appendix A we prove the Lipschitz condition and an auxiliary lemma.

2 Abstract Result: Proof of Theorem 1.1

We break the proof of Theorem 1.1 into two parts, stated as Lemmas 2.1 and 2.2:

Lemma 2.1. *Given $\varepsilon \in (0, 1)$, there exists $C_\varepsilon < \infty$ such that the following holds. Let u satisfy (1.1). Under assumptions (1.2) and (1.3), u and the associated function v satisfy:*

$$E(u(t)) - E(v(t)) \lesssim \exp(-2(1 - \varepsilon)t)(E(u(0)) - E(v(0))) + C_\varepsilon \delta^2. \quad (2.1)$$

Lemma 2.2. *Let u satisfy (1.1). Under assumptions (1.2) and (1.3), we have*

$$|u(t) - u(0)| \lesssim (E(u(0)) - E(v(0)))^{1/2} + \delta + \delta t. \quad (2.2)$$

In particular, for

$$t_1 := \max \left\{ 0, \log \left(\frac{E(u(0)) - E(v(0))}{\delta^2} \right) \right\} \quad (2.3)$$

we have

- For $t \in (0, t_1)$, $|u(t) - u(0)| \lesssim (E(u(0)) - E(v(0)))^{1/2}$.
- For $t \geq t_1$, $|u(t) - u(t_1)| \lesssim \delta + \delta(t - t_1)$.

The combination of Lemma 2.1 and (1.2) implies (1.4). Statement (2.2) of Lemma 2.2 implies (1.5). Thus, Theorem 1.1 is established as soon as we prove the lemmas.

Proof of Lemma 2.1. Let $e(t) := E(u(t)) - E(v(t))$. Recall that by the gradient flow dynamics and (1.2), we have

$$\frac{d}{dt} E(u(t)) = -|\nabla E|^2 \leq -2e(t).$$

Integrating from s to t , we deduce

$$e(t) - e(s) + 2 \int_s^t e(\tau) d\tau \leq E(v(s)) - E(v(t)). \quad (2.4)$$

Now we would like to take advantage of (1.3). To begin, we use the triangle inequality to estimate $|v(s) - v(t)|$:

$$\begin{aligned} |v(s) - v(t)| &\leq |v(s) - u(s)| + |v(t) - u(t)| + |u(t) - u(s)| \\ &\stackrel{(1.2)}{\leq} \sqrt{2e(s)} + \sqrt{2e(t)} + |u(t) - u(s)|. \end{aligned} \quad (2.5)$$

On the other hand, taking the weight $w \equiv 1$ in (1.8),

$$\begin{aligned} &|u(t) - u(s)| \\ &\leq \left((t-s)(E(u(s)) - E(u(t))) \right)^{1/2} \\ &= \left((t-s)(E(u(s)) - E(v(s)) + E(v(s)) - E(v(t)) + E(v(t)) - E(u(t))) \right)^{1/2} \\ &\leq \left((t-s)(e(s) + |E(v(s)) - E(v(t))|) \right)^{1/2}, \end{aligned} \quad (2.6)$$

where we have dropped the nonpositive term, $E(v(t)) - E(u(t))$. The combination of (1.3), (2.5), and (2.6) yields

$$\begin{aligned}
& |E(v(t)) - E(v(s))| \\
& \leq \delta |v(s) - v(t)| \\
& \leq \delta \left(\sqrt{2e(s)} + \sqrt{2e(t)} + \left((t-s)(e(s) + |E(v(s)) - E(v(t))|) \right)^{1/2} \right) \\
& \leq \delta \left(\sqrt{2e(s)} + \sqrt{2e(t)} + \sqrt{(t-s)e(s)} + \sqrt{(t-s)|E(v(s)) - E(v(t))|} \right).
\end{aligned}$$

By Young's inequality, this becomes

$$\begin{aligned}
& |E(v(t)) - E(v(s))| \\
& \leq 2\delta \left((\sqrt{2} + \sqrt{t-s})\sqrt{e(s)} + \sqrt{2e(t)} \right) + \delta^2(t-s). \tag{2.7}
\end{aligned}$$

Substituting (2.7) into (2.4),

$$\begin{aligned}
& e(t) - e(s) + 2 \int_s^t e(\tau) d\tau \\
& \leq 2\delta \left((\sqrt{2} + \sqrt{t-s})\sqrt{e(s)} + \sqrt{2e(t)} \right) + \delta^2(t-s),
\end{aligned}$$

which by Young's inequality may be expressed

$$(1 - \alpha)e(t) - (1 + \alpha)e(s) + 2 \int_s^t e(\tau) d\tau \leq C_\alpha \delta^2(1 + t - s)$$

for any fixed $\alpha \in (0, 1/2]$, where C_α is a constant depending only on α that may change from line to line.

Dividing through by $(1 + \alpha)$ and rearranging terms,

$$\frac{1 - \alpha}{1 + \alpha} e(t) \leq -\frac{2}{1 + \alpha} \int_s^t e(\tau) d\tau + e(s) + C_\alpha \delta^2(1 + t - s). \tag{2.8}$$

We will use (2.8) to derive a differential inequality.

Let $s \in (t-1, t)$ and use $(1 + \alpha)^{-1} \geq 1 - \alpha$ for $\alpha > 0$ to reexpress (2.8) as:

$$(1 - \alpha)^2 e(t) \leq -2(1 - \alpha) \int_s^t e(\tau) d\tau + e(s) + C_\alpha \delta^2. \tag{2.9}$$

As initial data for our inequality, we need the following: Notice that for $s = 0$ and $t \leq 1$, (2.9) implies

$$(1 - \alpha)^2 e(t) \leq e(0) + C_\alpha \delta^2,$$

and as a trivial consequence, we also have for any fixed $\tilde{\alpha} > 0$:

$$\begin{aligned} \int_0^1 \exp(-\tilde{\alpha}\tau)e(\tau) d\tau &\leq \frac{1}{(1-\alpha)^2}(e(0) + C_\alpha\delta^2) \\ &\lesssim e(0) + C_\alpha\delta^2, \end{aligned} \quad (2.10)$$

since for $\alpha \in (0, 1/2]$ we have $(1-\alpha)^{-2} \leq 4$.

Now let $\tilde{\alpha} := -\ln(1-\alpha)^2$ and rewrite (2.9), choosing $s = t-1$:

$$\exp(-\tilde{\alpha})e(t) \leq -2(1-\alpha) \int_{t-1}^t e(\tau) d\tau + e(t-1) + C_\alpha\delta^2. \quad (2.11)$$

Multiplying (2.11) by $\exp(-\tilde{\alpha}(t-1))$ gives

$$\begin{aligned} \exp(-\tilde{\alpha}t)e(t) &\leq -2(1-\alpha) \int_{t-1}^t \exp(-\tilde{\alpha}(t-1))e(\tau) d\tau \\ &\quad + \exp(-\tilde{\alpha}(t-1))(e(t-1) + C_\alpha\delta^2) \\ &\leq -2(1-\alpha) \int_{t-1}^t \exp(-\tilde{\alpha}\tau)e(\tau) d\tau \\ &\quad + \exp(-\tilde{\alpha}(t-1))(e(t-1) + C_\alpha\delta^2). \end{aligned} \quad (2.12)$$

Letting

$$F(t) := \int_{t-1}^t \exp(-\tilde{\alpha}\tau)e(\tau) d\tau, \quad (2.13)$$

equation (2.12) reads

$$F'(t) \leq -2(1-\alpha)F(t) + \exp(-\tilde{\alpha}(t-1))C_\alpha\delta^2,$$

so that

$$\begin{aligned} F(t) &\lesssim \exp(-2(1-\alpha)t)F(1) + \exp(-\tilde{\alpha}(t-1))C_\alpha\delta^2 \\ &\stackrel{(2.10)}{\lesssim} \exp(-2(1-\alpha)t)(e(0) + C_\alpha\delta^2) + \exp(-\tilde{\alpha}(t-1))C_\alpha\delta^2. \end{aligned} \quad (2.14)$$

Notice that by definition of $\tilde{\alpha}$ and the bound $\alpha \leq 1/2$, we have that $\exp(\tilde{\alpha}) \leq 4$, and in particular,

$$\exp(-\tilde{\alpha}(t-1)) \lesssim \exp(-\tilde{\alpha}t) \leq \exp(-\tilde{\alpha}\tau) \quad \tau \in (t-1, t). \quad (2.15)$$

Therefore, it follows that

$$\begin{aligned}
& \exp(-\tilde{\alpha}(t-1)) \int_{t-1}^t e(\tau) d\tau \\
& \stackrel{(2.15)}{\lesssim} \int_{t-1}^t \exp(-\tilde{\alpha}\tau) e(\tau) d\tau \\
& \stackrel{(2.13),(2.14)}{\lesssim} \exp(-2(1-\alpha)t)(e(0) + C_\alpha \delta^2) + \exp(-\tilde{\alpha}(t-1))C_\alpha \delta^2,
\end{aligned}$$

which implies

$$\begin{aligned}
\int_{t-1}^t e(\tau) d\tau & \lesssim \exp(-2(1-\alpha-\tilde{\alpha}/2)t)(e(0) + C_\alpha \delta^2) + C_\alpha \delta^2 \\
& \lesssim \exp(-2(1-\alpha-\tilde{\alpha}/2)t)e(0) + C_\alpha \delta^2
\end{aligned} \tag{2.16}$$

as long as

$$\alpha + \tilde{\alpha}/2 \leq 1,$$

which we will satisfy. We deduce from (2.16):

$$\inf_{(t-1,t)} e(\tau) \lesssim \exp(-2(1-\alpha-\tilde{\alpha}/2)t)e(0) + C_\alpha \delta^2. \tag{2.17}$$

This is enough to bound the supremum: Suppose the supremum on $[t-1, t]$ is achieved at t_{max} and the infimum at t_{min} . From (2.8) with $t = t_{max}$ and $s = t_{min}$,

$$\begin{aligned}
e(t_{max}) & \lesssim - \int_{t_{min}}^{t_{max}} e(\tau) d\tau + e(t_{min}) + C_\alpha \delta^2 \\
& \leq \int_{t-1}^t e(\tau) d\tau + e(t_{min}) + C_\alpha \delta^2 \\
& \stackrel{(2.16),(2.17)}{\lesssim} \exp(-2(1-\alpha-\tilde{\alpha}/2)t)e(0) + C_\alpha \delta^2.
\end{aligned}$$

Defining $\varepsilon := \alpha + \tilde{\alpha}/2 = \alpha - \log(1-\alpha)$ completes the proof of (2.1). (Notice that $f(x) = x - \log(1-x)$ is continuous and monotone increasing on $(0, 1/2]$ with $f(0) = 0$, $f(1/2) > 1$, so we can satisfy (2.1) for any $\varepsilon \in (0, 1)$.) \square

Proof of Lemma 2.2. For simplicity, we now set $\varepsilon = 1/2$ in Lemma 2.1. The value of t_1 has been chosen precisely so that

$$\exp(-2(1-\varepsilon)t_1)e(0) = \exp(-t_1)e(0) = \delta^2,$$

where we recall the definition $e(t) := E(u(t)) - E(v(t))$. Notice that:

$$0 \leq t \leq t_1 \quad \Rightarrow \quad \exp(-t)e(0) \geq \delta^2, \quad (2.18)$$

$$t_1 \leq t \quad \Rightarrow \quad \exp(-t)e(0) \leq \delta^2. \quad (2.19)$$

We begin by considering $t \geq t_1$, where by (1.4) and (2.19),

$$\frac{1}{2}|u(t) - v(t)|^2 + e(t) \lesssim \delta^2. \quad (2.20)$$

Therefore, combining equations (2.5) and (2.6) from the proof of Lemma 2.1, we have:

$$\begin{aligned} & |v(t) - v(t_1)| \\ & \leq \sqrt{2e(t)} + \sqrt{2e(t_1)} + \left((t - t_1)(e(t_1) + |E(v(t_1)) - E(v(t))|) \right)^{1/2} \\ & \stackrel{(1.3)}{\leq} \sqrt{e(t_1)}(\sqrt{2} + \sqrt{t - t_1}) + \sqrt{2e(t)} + \delta^{1/2}\sqrt{t - t_1}|v(t) - v(t_1)|^{1/2} \\ & \stackrel{(2.20)}{\lesssim} \delta(1 + \sqrt{t - t_1}) + \delta^{1/2}\sqrt{t - t_1}|v(t) - v(t_1)|^{1/2}. \end{aligned}$$

By Young's inequality, this implies

$$|v(t) - v(t_1)| \lesssim \delta(1 + t - t_1).$$

Together with the triangle inequality and (2.20), this yields

$$|u(t) - u(t_1)| \lesssim \delta(1 + t - t_1) \quad \text{for } t \geq t_1.$$

We now turn to the case $0 \leq t \leq t_1$. (We may assume without loss that $t_1 > 0$ since if $t_1 = 0$, we are already finished.) Here we use a different weight in the gradient flow inequality (1.8):

$$w(s) := \exp(s/4).$$

This yields the inequality

$$\begin{aligned}
& |u(t) - u(0)|^2 \\
& \lesssim E(u(0)) - \exp(t/4)E(u(t)) + \int_0^t \exp(s/4)E(u(s))ds \\
& = E(u(0)) - E(v(0)) + E(v(0)) - \exp(t/4)E(v(t)) \\
& \quad - \exp(t/4)(E(u(t)) - E(v(t))) \\
& \quad + \int_0^t \exp(s/4)(E(u(s)) - E(v(s))) ds \\
& \quad + \int_0^t \exp(s/4)E(v(s)) ds \\
& \stackrel{(1.4)}{\leq} E(u(0)) - E(v(0)) + E(v(0)) - \exp(t/4)E(v(t)) \\
& \quad + \int_0^t \exp(s/4)(E(u(s)) - E(v(s))) ds \\
& \quad + \int_0^t \exp(s/4)E(v(s)) ds. \tag{2.21}
\end{aligned}$$

Without loss, we may assume that $E(v(0)) = 0$. Observe then that by (1.3),

$$|E(v(t))| \leq \delta|v(t) - v(0)|. \tag{2.22}$$

The combination of (2.21), (2.22), and Lemma 2.1 with $\varepsilon = 1/2$ gives for $t \leq t_1$:

$$\begin{aligned}
& |u(t) - u(0)|^2 \\
& \lesssim e(0) + \delta \exp(t/4)|v(t) - v(0)| + \int_0^t \exp(s/4) \left(e(0) \exp(-s) + \delta^2 \right) ds \\
& \quad + \delta \int_0^t \exp(s/4)|v(s) - v(0)| ds \\
& \stackrel{(2.18)}{\lesssim} e(0) + \delta \exp(t/4)|v(t) - v(0)| + \int_0^t \exp(s/4)e(0) \exp(-s) ds \\
& \quad + \delta \int_0^t \exp(s/4)|v(s) - v(0)| ds \\
& \lesssim e(0) + \delta \exp(t_1/4) \sup_{t \leq t_1} |v(t) - v(0)|. \tag{2.23}
\end{aligned}$$

Using

$$|v(t) - v(0)| \leq |v(t) - u(t)| + |v(0) - u(0)| + |u(t) - u(0)|,$$

we deduce, after another application of (1.4), that

$$\begin{aligned} \sup_{t \leq t_1} |v(t) - v(0)| &\lesssim \sqrt{e(0)} + \delta + \sup_{t \leq t_1} |u(t) - u(0)| \\ &\stackrel{(2.18)}{\lesssim} \sqrt{e(0)} + \sup_{t \leq t_1} |u(t) - u(0)|. \end{aligned} \quad (2.24)$$

Taking the supremum on the left-hand side of (2.23), substituting (2.24), and applying Young's inequality one more time, we arrive at:

$$\begin{aligned} \sup_{t \leq t_1} |u(t) - u(0)|^2 &\lesssim e(0) + \delta \exp(t_1/4) \sqrt{e(0)} + \delta^2 \exp(t_1/2) \\ &\lesssim e(0) + \delta^2 \exp(t_1/2) \\ &\stackrel{(2.3)}{=} e(0) + \delta \sqrt{e(0)} \\ &\stackrel{(2.18)}{\lesssim} e(0). \end{aligned}$$

□

3 Application to coarsening in Allen-Cahn

In Subsection 3.1, we apply the abstract result to prove Theorem 1.2. In order to invoke Theorem 1.1, we need to show that Assumptions (i) and (ii) are satisfied. Most of the work lies in proving the energy-energy-dissipation relationship. We prove a linearized version in Subsection 3.2. Then in Subsection 3.3, we improve from the linearized estimates to the nonlinear estimates, proving Proposition 1.1. The ingredients for Assumption (ii) (Lemmas 3.1 and 3.2) are proved in Appendix A.

As discussed in Remark 3, we will use the fact that for the Allen-Cahn equation, the number of zeros can only decrease in time [A], so that the sign of u in-between two adjacent zeros is well-defined and constant for as long as the zeros exist.

3.1 Proof of Theorem 1.2

Before we begin, we give two lemmas that we will need for the Lipschitz condition. First, it is convenient to introduce:

Notation 1. *Given a periodic function $v : [0, L] \rightarrow \mathbb{R}$ with two simple zeros, let x_v denote the zero at which v changes from negative to positive, and y_v the one at which it changes back to negative. Moreover, let ℓ_v denote the minimal distance between zeros of v , taking into account the periodicity.*

The basic Lipschitz property is formulated:

Lemma 3.1. *Let v and w be periodic, energy-optimal profiles with two simple zeros. There exist constants $\ell_2, C_2 < \infty$ such that if $\min\{\ell_v, \ell_w\} \geq \ell_2$, then*

$$|E(v) - E(w)| \leq \delta (|x_w - x_v| + |y_w - y_v|), \quad (3.1)$$

with $\delta := C_2 \exp(-\sqrt{G''(1)} \min\{\ell_v, \ell_w\})$.

The estimate in (3.1) is formulated in terms of the positions of the zeros, the natural distance on the space of energy optimal profiles. We will need to be able to go back and forth between this distance and the L^2 -distance. Lemma 3.2 serves this purpose:

Lemma 3.2. *There exists a constant $\ell_4 < \infty$ with the following property. Let v and w be periodic, energy-optimal profiles with two simple zeros. Assume that $\min\{\ell_v, \ell_w\} \geq \ell_4$ and*

$$|x_v - x_w| + |y_v - y_w| \leq \ell_4^{-1} \min\{\ell_v, \ell_w\}.$$

Then we have

$$\begin{aligned} & |x_v - x_w| + |y_v - y_w| \\ & \lesssim \|v - w\|_{L^2([0,L])}^2 + \|v - w\|_{L^2([0,L])} \end{aligned} \quad (3.2)$$

$$\lesssim \|v - w\|_{L^2([0,L])}^2 + 1, \quad (3.3)$$

and

$$\|v - w\|_{L^2([0,L])} \lesssim |x_v - x_w| + |y_v - y_w| + 1. \quad (3.4)$$

We recall the notation from the statement of Theorem 1.2: $x(t), y(t)$ denote the location of the zeros at time t and $\ell(t)$ denotes the minimal distance between zeros. Also, for convenience we define

$$\Delta x(T) := \sup_{t \leq T} (|x(t) - x(0)| + |y(t) - y(0)|).$$

We now combine Lemmas 3.1 and 3.2 with Proposition 1.1 to derive the following result from Theorem 1.1:

Lemma 3.3. *Let G satisfy (1.11). There exist constants $\ell_* < \infty, C < \infty$ with the property that if u solves (1.13) and satisfies (1.14), then if $T > 0$ is such that*

$$\Delta x(T) \leq \min \left\{ \frac{1}{2}, \frac{1}{1 + C_4} \right\} \ell(0), \quad (3.5)$$

then we have for all $t \leq T$ that the energy restricted to the slow manifold is Lipschitz with constant

$$\delta(T) := C \left(\sup_{t \leq T} \left(\int_0^L (v(t) - v(0))^2 dx \right)^{1/2} + 1 \right) \exp \left(- \sqrt{G''(1)} \inf_{t \leq T} \ell(t) \right),$$

and for arbitrary $\varepsilon \in (0, 1)$, the solution u and the associated energy optimal profile v satisfy

$$\begin{aligned} & \|u(t) - v(t)\|_{H^1([0, L])} + (E(u(t)) - E(v(t)))^{1/2} \\ & \lesssim \exp(-(1 - \varepsilon)t/C_1) (E(u(0)) - E(v(0)))^{1/2} + C_\varepsilon \delta, \end{aligned} \quad (3.6)$$

and

$$\|u(t) - u(s)\|_{H^1([0, L])} \lesssim (E(u(s)) - E(v(s)))^{1/2} + \delta(t - s + 1) \quad (3.7)$$

for any $0 < s < t$.

Proof of Lemma 3.3. Let $\ell_* < \infty$ be a constant to be specified later. We want to deduce (3.6) and (3.7) from Theorem 1.1. Thus, the main task is to verify that Assumptions (i) and (ii) hold for the evolution on $(0, T]$.

To use Proposition 1.1 to verify Assumption (i), we need to rescale energy and time so that

$$\tilde{E} = C_1 E \quad \tilde{t} = t/C_1.$$

According to the proposition, if

$$E \text{ between adjacent zeros is less than } (G(0)/2)\ell_1, \quad (3.8)$$

$$\inf_{t \leq T} \ell(t) \geq \ell_1, \quad (3.9)$$

then in the rescaled variables, u and v satisfy Assumption (i).

On the other hand, according to Lemma 3.1 and (3.2), if

$$\inf_{t \leq T} \ell(t) \geq \max\{\ell_2, \ell_4\}, \quad (3.10)$$

$$\sup_{t \leq T} (|x(t) - x(0)| + |y(t) - y(0)|) \leq \ell_4^{-1} \inf_{t \leq T} \ell(t), \quad (3.11)$$

then there exists a constant $C < \infty$ such that the rescaled energy satisfies Assumption (ii) with the $H^1([0, L])$ norm and

$$\delta(T) := C \left(\sup_{t \leq T} \left(\int_0^L (v(t) - v(0))^2 dx \right)^{1/2} + 1 \right) \exp \left(- \sqrt{G''(1)} \inf_{t \leq T} \ell(t) \right).$$

We will now check that (3.8)-(3.11) are satisfied if ℓ_* is chosen such that

$$\ell_* \geq 2 \max\{\ell_1, \ell_2, \ell_4\}, \quad (3.12)$$

and T satisfies (3.5). Indeed, (3.8) follows from the assumption

$$E(u_0) \leq 4c_0,$$

(cf. (1.14)) since the energy between zeros is bounded by the total energy, the energy is a decreasing function of time, and we may assume without loss that $\ell_1 \geq 8c_0/G(0)$. Next we claim that

$$\inf_{t \leq T} \ell(t) \geq \ell(0) - \Delta x(T), \quad (3.13)$$

which will follow from the stronger statement:

$$\sup_{t \leq T} |\ell(0) - \ell(t)| \leq \Delta x(T). \quad (3.14)$$

To see (3.14) we observe that

$$\begin{aligned} \ell(t) - \ell(0) &= \min\{|(x(t), y(t))|, |(y(t), x(t))|\} - \min\{|(x(0), y(0))|, |(y(0), x(0))|\} \\ &\leq \max\{|(x(t), y(t))| - |(x(0), y(0))|, |(y(t), x(t))| - |(y(0), x(0))|\} \\ &\leq \max\{|(x(t), y(t)) \cap (y(0), x(0))|, |(y(t), x(t)) \cap (x(0), y(0))|\} \\ &\leq |x(t) - x(0)| + |y(t) - y(0)|. \end{aligned}$$

Using symmetry and taking the supremum, (3.14) follows.

Using (3.13), we compute:

$$\inf_{t \leq T} \ell(t) \geq \ell(0) - \Delta x(T) \stackrel{(3.5)}{\geq} \frac{1}{2} \ell(0) \stackrel{(3.12)}{\geq} \max\{\ell_1, \ell_2, \ell_4\},$$

which verifies (3.9) and (3.10). Similarly, for (3.11) we observe:

$$\inf_{t \leq T} \ell(t) \geq \ell(0) - \Delta x(T) \stackrel{(3.5)}{\geq} \ell_4 \Delta x(T).$$

Thus, we may invoke Theorem 1.1. Rewriting (1.4) and (1.5) in the original variables gives (3.6) and (3.7). \square

We are now ready to prove Theorem 1.2. It will follow from Lemma 3.3 and a buckling argument.

Proof of Theorem 1.2. Let $\ell_* < \infty$ be a constant greater than or equal to the constant from Lemma 3.3; we will specify further requirements for ℓ_* later. Our first observation is that for $\delta(T)$ defined in Lemma 3.3, we have

$$\delta(T) \stackrel{(3.4)}{\lesssim} (\Delta x(T) + 1) \exp\left(-\sqrt{G''(1)} \inf_{t \leq T} \ell(t)\right). \quad (3.15)$$

We require T to satisfy (3.5) as well as

$$\delta(T) \leq 1. \quad (3.16)$$

Now let

$$T^* := \min\left\{\epsilon \exp\left(\sqrt{G''(1)}\ell(0)\right), T\right\}, \quad (3.17)$$

where $\epsilon > 0$ is a small constant to be specified later. The goal is to show that it is possible to choose $T \geq T^*$.

From (3.7) and (1.14), we have for $t \leq T^*$:

$$\begin{aligned} & \left(\int_0^L (u(t) - u(0))^2 dx\right)^{1/2} \\ & \lesssim 1 + \delta + \delta T^* \\ & \stackrel{(3.16)}{\lesssim} 1 + \delta T^* \\ & \stackrel{(3.15), (3.17)}{\lesssim} 1 + \epsilon(\Delta x(T^*) + 1) \exp\left(\sqrt{G''(1)} \sup_{t \leq T^*} |\ell(0) - \ell(t)|\right). \end{aligned} \quad (3.18)$$

On the one hand, we recall (3.14). On the other hand, we claim that

$$(\Delta x(T^*))^{1/2} \lesssim \sup_{t \leq T^*} \left(\int_0^L (u(t) - u(0))^2 dx\right)^{1/2} + 1. \quad (3.19)$$

This is a consequence of (3.3) together with

$$\begin{aligned} & \int_0^L (v(t) - v(0))^2 dx \\ & \lesssim \int_0^L (u(t) - u(0))^2 dx + \int_0^L (u(0) - v(0))^2 dx + \int_0^L (u(t) - v(t))^2 dx \\ & \stackrel{(1.20)}{\lesssim} \int_0^L (u(t) - u(0))^2 dx + E(u(0)) - E(v(0)) + E(u(t)) - E(v(t)) \\ & \leq \int_0^L (u(t) - u(0))^2 dx + 2E(u(0)) - E(v(0)) - E(v(t)) \\ & \stackrel{(1.14)}{\lesssim} \int_0^L (u(t) - u(0))^2 dx + 1. \end{aligned}$$

The combination of (3.19), (3.18), and (3.14) gives:

$$(\Delta x(T^*))^{1/2} \lesssim 1 + \epsilon(\Delta x(T^*) + 1) \exp\left(\sqrt{G''(1)}\Delta x(T^*)\right).$$

By continuity of $\Delta x(t)$ and $\Delta x(0) = 0$, choosing ϵ sufficiently small implies

$$\Delta x(T^*) \leq C. \quad (3.20)$$

We are now ready to fix ℓ_* . First, we remark that

$$\inf_{t \leq T} \ell(t) \geq \ell(0) - \Delta x(T) \geq \ell(0) - C, \quad (3.21)$$

and observe:

$$\begin{aligned} \delta(T^*) &\stackrel{(3.15)}{\lesssim} (\Delta x(T^*) + 1) \exp(-\sqrt{G''(1)}\ell(t)) \\ &\stackrel{(3.20),(3.21)}{\lesssim} \exp(-\sqrt{G''(1)}\ell(0)), \end{aligned} \quad (3.22)$$

$$(3.23)$$

and in particular,

$$\delta(T^*) \lesssim \exp(-\sqrt{G''(1)}\ell_*).$$

Therefore by choosing ℓ_* sufficiently large, we can guarantee

$$\delta(T^*) \leq 1. \quad (3.24)$$

Second, we require

$$\ell_* \geq C / \min\left\{\frac{1}{2}, \frac{1}{1+C_4}\right\}, \quad (3.25)$$

so that

$$\begin{aligned} \Delta x(T^*) &\stackrel{(3.20)}{\leq} C \stackrel{(3.25)}{\leq} \min\left\{\frac{1}{2}, \frac{1}{1+C_4}\right\} \ell_* \\ &\leq \min\left\{\frac{1}{2}, \frac{1}{1+C_4}\right\} \ell(0). \end{aligned} \quad (3.26)$$

Together (3.24) and (3.26) imply that we can choose $T \geq T^*$, so that

$$T^* = \epsilon \exp(\sqrt{G''(1)}\ell(0)).$$

Thus, for all $t \leq \epsilon \exp(\sqrt{G''(1)}\ell(0))$, we have

$$|x(t) - x(0)| + |y(t) - y(0)| \lesssim 1.$$

In addition, (3.22) and (3.6) together with

$$E(u(0)) - E(v(0)) \lesssim 1$$

imply that by choosing $t/\ell(0)$ sufficiently large, we have

$$\|u(t) - v(t)\|_{H^1([0,L])} \lesssim \delta \stackrel{(3.22)}{\lesssim} \exp(-\sqrt{G''(1)}\ell(0)).$$

□

3.2 Linear energy-energy-dissipation estimates

The goal of this subsection is to develop the estimate:

Proposition 3.1 (Finite system). *There exist constants $\ell_3, C_3 < \infty$ such that for all $\ell \geq \ell_3$, for all smooth f with $f(0) = f(\ell) = 0$,*

$$\|f\|_{H^1([0,\ell])}^2 \leq C_3 \int_0^\ell \left(f_x^2 + G''(v) f^2 \right) dx, \quad (3.27)$$

$$\|f\|_{H^2([0,\ell])}^2 \leq C_3 \int_0^\ell \left(-f_{xx} + G''(v) f \right)^2 dx, \quad (3.28)$$

where v solves

$$\begin{aligned} -v_{xx} + G'(v) &= 0 & x \in (0, \ell), \\ v &> 0 & x \in (0, \ell), \\ v &= 0 & x \in \{0, \ell\}. \end{aligned} \quad (3.29)$$

To prove the proposition, we first state and prove a related result for the large-system limit.

Proposition 3.2 (Infinite system). *There exists a constant $C'_3 > 0$ such that if the smooth function v is the solution of*

$$\begin{aligned} -v_{xx} + G'(v) &= 0 & x \in \mathbb{R}, \\ v &\rightarrow \pm 1 & x \rightarrow \pm\infty, \end{aligned} \quad (3.30)$$

and if $f \in H^1(\mathbb{R})$ satisfies:

$$(i) f(0) = 0 \quad \text{or} \quad (ii) \int_{\mathbb{R}} f v_x dx = 0,$$

then

$$\|f\|_{L^2(\mathbb{R})}^2 \leq C'_3 \int_{\mathbb{R}} \left(f_x^2 + G''(v) f^2 \right) dx.$$

Proof of Proposition 3.2. We begin by introducing two lemmas.

Lemma 3.4. *Let v be as in Proposition 3.2. For all $f \in H^1(\mathbb{R})$,*

$$\int_{\mathbb{R}} \left(f_x^2 + G''(v)f^2 \right) dx \geq 0.$$

Proof. This is a consequence of the well-known fact that v minimizes the energy over functions satisfying the boundary conditions. To see that v is the minimizer, one observes that (3.30) implies:

$$\partial_x \left(\frac{v_x^2}{2} - G(v) \right) = 0,$$

and from the boundary conditions, one may conclude equipartition of energy, meaning that

$$\frac{v_x^2}{2} - G(v) = 0. \tag{3.31}$$

This implies

$$E(v) = \int_{\mathbb{R}} |v_x \sqrt{2G(v)}| dx = \int_{-1}^1 \sqrt{2G(v)} dv.$$

At the same time, the “Modica-Mortola calculation” shows that the energy of any admissible function w is at least this big:

$$\frac{1}{2} \int_{\mathbb{R}} (w_x^2 + 2G(w)) dx \geq \int_{\mathbb{R}} |w_x \sqrt{G(w)}| dx \geq \int_{-1}^1 \sqrt{2G(w)} dw.$$

□

Lemma 3.5. *For all $f \in C^2(\mathbb{R}) \cap H^1(\mathbb{R})$,*

$$-f_{xx} + G''(v)f = 0 \quad \Rightarrow \quad f = \alpha v_x \quad \text{for some } \alpha \in \mathbb{R}.$$

Proof. Let $\phi := v_x$. We know by the maximum principle that $v \in (-1, 1)$ on \mathbb{R} , so $G(v) > 0$ and by (3.31), ϕ is of one sign. Furthermore, ϕ satisfies $-\phi_{xx} + G''(v)\phi = 0$. Suppose f is another solution. Then we have

$$\frac{d}{dx}(f'\phi - \phi'f) = 0.$$

Since f and $\phi \in H^1(\mathbb{R})$, $f'\phi - \phi'f = 0$. Since ϕ is nonzero, we can consider the ratio f/ϕ , and we see that

$$\frac{d}{dx} \left(\frac{f}{\phi} \right) = \frac{f'\phi - f\phi'}{\phi^2} = 0.$$

□

The proof of the proposition now follows by contradiction. Assume that there exists an H^1 -sequence $\{f_n\}_{n=1}^\infty$ with $f_n(0) = 0$, $\|f_n\|_{L^2(\mathbb{R})} = 1$, and

$$\int_{\mathbb{R}} \left(f_{n,x}^2 + G''(v)f_n^2 \right) dx < \frac{1}{n} \int_{\mathbb{R}} f_n^2 dx. \quad (3.32)$$

This implies that f_n is uniformly bounded in $H^1(\mathbb{R})$, so there is a subsequence converging weakly in H^1 and strongly in L^2 to a limit f , and by lower semi-continuity,

$$\int_{\mathbb{R}} (f_x^2 + G''(v)f^2) dx \leq 0.$$

Therefore, by Lemma 3.4, f minimizes this functional and thus satisfies the Euler-Lagrange equation

$$-f_{xx} + G''(v)f = 0.$$

From here we can conclude that $f \in C^2(\mathbb{R})$, so Lemma 3.5 implies $f = \alpha v_x$. But $f(0) = 0$ and $v_x(0) \neq 0$ implies $\alpha = 0$, so $f \equiv 0$. We now show that this leads to a contradiction.

First of all, there exists an $X < \infty$ such that $G''(v(x)) \geq \frac{1}{2}G''(1) > 0$ on $\mathbb{R} \setminus (-X, X)$. Because an H^1 bound gives a Hölder bound in one dimension, the Arzela-Ascoli theorem implies that f_n converges locally uniformly (up to a subsequence). Thus, we may assume that $f_n \rightarrow 0$ uniformly on $(-X, X)$, and by the L^2 -convergence, $\int_{\mathbb{R} \setminus (-X, X)} f_n^2 dx \rightarrow 1$. But then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}} G''(v)f_n^2 dx \\ & \geq -\sup_{\mathbb{R}} |G''(v(x))| \lim_{n \rightarrow \infty} \int_{-X}^X f_n^2 dx + \frac{1}{2}G''(1) \lim_{n \rightarrow \infty} \int_{\mathbb{R} \setminus (-X, X)} f_n^2 dx \\ & = 0 + \frac{1}{2}G''(1) \\ & > 0. \end{aligned}$$

This contradicts (3.32) and proves the proposition under condition (i). The proof under condition (ii) is similar; assume the existence of an H^1 -sequence $\{f_n\}_{n=1}^\infty$ with $\int_{\mathbb{R}} f_n v_x dx = 0$, $\|f_n\|_{L^2(\mathbb{R})} = 1$, and (3.32). One may conclude, as above, convergence to a limit f and, by Lemma 3.5, that $f = \alpha v_x$. But then

$$0 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n v_x dx = \int_{\mathbb{R}} f v_x dx = \alpha \int_{\mathbb{R}} v_x^2 dx \Rightarrow \alpha = 0.$$

This implies $f \equiv 0$, which leads to a contradiction, as in the first case. \square

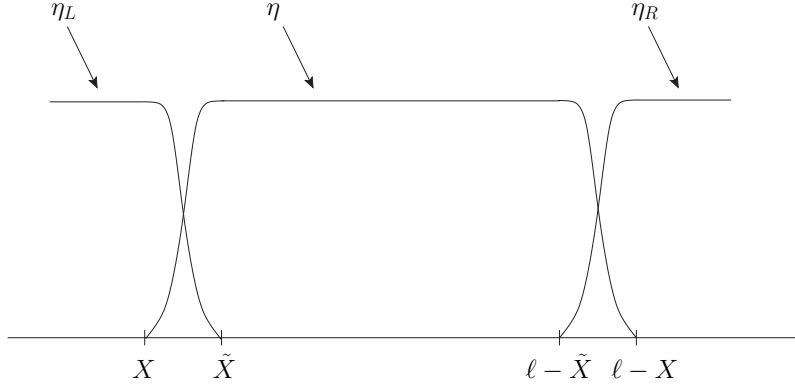


Figure 3: The partition of unity.

Proof of Proposition 3.1. We turn now to the proof of the main proposition. We begin by showing

$$\int_0^\ell f^2 dx \leq C_3 \int_0^\ell \left(f_x^2 + G''(v) f^2 \right) dx, \quad (3.33)$$

and then complete the proof by bootstrapping.

Let v_ℓ denote the solution of (3.29) on $[0, \ell]$. The idea is to split up the domain, using on one part that $G''(v_\ell)$ is bounded away from zero, and using on the other part that $v_\ell \rightarrow v_\infty$ uniformly as $\ell \rightarrow \infty$, allowing us to take advantage of Proposition 3.2. (Here, v_∞ represents the solution of (3.30) with $v_\infty(0) = 0$.) To implement the idea, we use a partition of unity. We split the domain in the following way. (See Figure 3.) Fix X such that

$$G''(v_\infty(x)) > \frac{G''(1)}{2} =: c_* \text{ for } x > X.$$

The functions v_ℓ converge in the sense:

$$v_\infty(x) = \lim_{\ell \rightarrow \infty} \begin{cases} v_\ell(x) & x \in [0, \ell/2), \\ v_\ell(\ell - x) & x \in (\ell/2, \ell], \end{cases} \quad (3.34)$$

with the convergence uniform on bounded sets. Also, v_ℓ is symmetric about $\ell/2$, and monotone on $(0, \ell/2)$. Thus, there exists an $\ell_3(X)$ such that for all $\ell \geq \ell_3$, we have that

$$G''(v_\ell(x)) > \frac{c_*}{2} \text{ for } x \in (X, \ell - X). \quad (3.35)$$

Define $c_{**} := \min\{c_*, 1/C'_3\}$ (where C'_3 is from Proposition 3.2) and

$$\tilde{X} := X + \left(\frac{16}{c_{**}}\right)^{1/2}, \quad (3.36)$$

a choice that will become clear later. Finally, in view of (3.34), we may assume that ℓ_3 is so large that for all $\ell \geq \ell_3$ we have:

$$G''(v_\ell(x)) \geq G''(v_\infty(x)) - \frac{1}{2C'_3} \quad \text{for } x \in [0, \tilde{X}]. \quad (3.37)$$

Choose a cut-off function $\eta \in C^\infty([0, \ell])$ with $|\eta| \leq 1$,

$$\eta = \begin{cases} 1 & x \in [\tilde{X}, \ell - \tilde{X}] \\ 0 & x \in [0, X] \cap [\ell - X, \ell], \end{cases}$$

and

$$|\eta_{xx}| \leq \frac{2}{(\tilde{X} - X)^2}. \quad (3.38)$$

Define also the “left” and “right” functions:

$$\eta_L := \begin{cases} 1 - \eta & x \in [0, \tilde{X}] \\ 0 & x \in (\tilde{X}, \ell], \end{cases} \quad \eta_R := \begin{cases} 0 & x \in [0, \ell - \tilde{X}], \\ 1 - \eta & x \in [\ell - \tilde{X}, \ell]. \end{cases}$$

Then $\{\eta_L, \eta, \eta_R\}$ is a smooth partition of unity for $[0, \ell]$, which we will use to prove (3.33). By (3.35),

$$\int_0^\ell \left(((\eta f)_x)^2 + G''(v_\ell)(\eta f)^2 \right) dx \geq \frac{c_*}{2} \int_0^\ell (\eta f)^2 dx. \quad (3.39)$$

By (3.37) and Proposition 3.2,

$$\begin{aligned} & \int_0^\ell \left(((\eta_L f)_x)^2 + G''(v_\ell)(\eta_L f)^2 \right) dx \\ & \geq \int_0^\ell \left(((\eta_L f)_x)^2 + G''(v_\infty)(\eta_L f)^2 - \frac{1}{2C'_3}(\eta_L f)^2 \right) dx \\ & \geq \frac{1}{2C'_3} \int_0^\ell (\eta_L f)^2 dx. \end{aligned} \quad (3.40)$$

Similarly,

$$\int_0^\ell \left(((\eta_R f)_x)^2 + G''(v_\ell)(\eta_R f)^2 \right) dx \geq \frac{1}{2C'_3} \int_0^\ell (\eta_R f)^2 dx. \quad (3.41)$$

We add (3.39), (3.40), and (3.41). For the right-hand side, we claim that

$$\begin{aligned}
\frac{1}{2C_3'} \int_0^\ell ((\eta_L f)^2 + (\eta_R f)^2) dx + \frac{c_*}{2} \int_0^\ell (\eta f)^2 dx \\
\geq \frac{c_{**}}{2} \int_0^\ell f^2 (\eta_L^2 + \eta^2 + \eta_R^2) dx \\
\geq \frac{c_{**}}{4} \int_0^\ell f^2 dx. \tag{3.42}
\end{aligned}$$

Clearly, (3.42) is true wherever $\eta = 1$ or $\eta = 0$. We check what happens on $[X, \tilde{X}]$. By the definition of η_L and the fact that $\eta(x) \in [0, 1]$,

$$\eta^2 + \eta_L^2 = \eta^2 + (1 - \eta)^2 = 1 - 2\eta(1 - \eta) \geq \frac{1}{2},$$

and similarly on $[\ell - \tilde{X}, \ell - X]$, so (3.42) is true. Next, we claim that the left-hand side of the inequality satisfies

$$\begin{aligned}
\int_0^\ell \left(((\eta_L f)_x)^2 + ((\eta f)_x)^2 + ((\eta_R f)_x)^2 + (\eta_L^2 + \eta^2 + \eta_R^2) G''(v_\ell) f^2 \right) dx \\
\leq \int_0^\ell (f_x^2 + G''(v_\ell) f^2) dx + \frac{c_{**}}{8} \int_0^\ell f^2 dx, \tag{3.43}
\end{aligned}$$

which, together with (3.42), proves (3.33) (with $C_3 := 8/c_{**}$). Again, the equality is obvious when $\eta = 1$ or $\eta = 0$, and we check what happens on $[X, \tilde{X}]$ and $[\ell - \tilde{X}, \ell - X]$. There, we have $G''(v_\ell) \geq 0$, so that $\eta_L^2 + \eta^2 + \eta_R^2 \leq 1$ implies for the zeroth order terms:

$$\int_{[X, \tilde{X}] \cup [\ell - \tilde{X}, \ell - X]} G''(v_\ell) f^2 (\eta_L^2 + \eta^2 + \eta_R^2) dx \leq \int_{[X, \tilde{X}] \cup [\ell - \tilde{X}, \ell - X]} G''(v_\ell) f^2 dx.$$

For the derivative terms, we use the following identity, which comes from expanding the square and integrating by parts:

$$\int_0^\ell ((wf)_x)^2 dx = \int_0^\ell (w^2 f_x^2 - w w_{xx} f^2) dx.$$

Adding the three contributions and recalling the definition of η_L and η_R ,

$$\begin{aligned}
& \int_0^\ell ((\eta_L f)_x)^2 + ((\eta f)_x)^2 + ((\eta_R f)_x)^2 dx \\
&= \int_0^\ell (\eta_L^2 + \eta^2 + \eta_R^2) f_x^2 - \eta \eta_{xx} f^2 - \eta_L \eta_{\ell,xx} f^2 - \eta_R \eta_{r,xx} f^2 dx \\
&= \int_0^\ell (\eta_L^2 + \eta^2 + \eta_R^2) f_x^2 - \eta \eta_{xx} f^2 - (1 - \eta)(-\eta_{xx}) f^2 dx \\
&= \int_0^\ell (\eta_L^2 + \eta^2 + \eta_R^2) f_x^2 + (1 - 2\eta) \eta_{xx} f^2 dx \\
&\leq \int_0^\ell f_x^2 + |\eta_{xx}| f^2 dx.
\end{aligned}$$

Recall that by (3.36) and (3.38), we have chosen η and \tilde{X} such that

$$|\eta_{xx}| \leq \frac{2}{(\tilde{X} - X)^2} = \frac{c_{**}}{8}.$$

Therefore, (3.43) holds and the proof of (3.33) is complete.

It is not hard to complete the proof of (3.27) by bootstrapping from (3.33) to the H^1 -norm. Letting $\delta > 0$ be such that

$$\delta \leq \frac{1}{2C_3 \sup_{[0,1]} |G''(y)|}, \quad (3.44)$$

we have:

$$\begin{aligned}
(1 + \delta) & \int_0^\ell (f_x^2 + G''(v_\ell) f^2) dx \\
&= \int_0^\ell (f_x^2 + G''(v_\ell) f^2) dx + \delta \int_0^\ell f_x^2 dx + \delta \int_0^\ell G''(v_\ell) f^2 dx \\
&\geq \frac{1}{C_3} \int_0^\ell f^2 dx + \delta \int_0^\ell f_x^2 dx - \frac{1}{2C_3} \int_0^\ell f^2 dx \\
&\geq \min \left\{ \frac{1}{2C_3}, \delta \right\} \int_0^\ell (f^2 + f_x^2) dx,
\end{aligned}$$

which proves (3.27) with C_3 appropriately redefined.

Turning to (3.28), it is convenient to introduce the linear operator \mathcal{L} defined by $\mathcal{L}f := f_{xx} - G''(v_\ell)f$, whereby (3.28) reads:

$$\|f\|_{H^2([0,\ell])}^2 \leq C_3 \|\mathcal{L}f\|_{L^2([0,\ell])}^2.$$

Let (\cdot, \cdot) denote the L^2 inner product on $[0, \ell]$. First, by (3.27) and Hölder's inequality, we have

$$\|f\|_{L^2([0, \ell])}^2 \leq C_3 |(f, \mathcal{L}f)| \leq C_3 \|f\|_{L^2([0, \ell])} \|\mathcal{L}f\|_{L^2([0, \ell])}, \quad (3.45)$$

which implies

$$\|f\|_{L^2([0, \ell])} \leq C_3 \|\mathcal{L}f\|_{L^2([0, \ell])}.$$

Next, substituting for $\|f\|_{L^2([0, \ell])}$ in the right-hand side of (3.45), we observe

$$|(f, \mathcal{L}f)| \leq C_3 \|\mathcal{L}f\|_{L^2([0, \ell])}^2.$$

Invoking (3.27) again, we conclude:

$$\|f\|_{H^1([0, \ell])}^2 \leq C_3^2 \|\mathcal{L}f\|_{L^2([0, \ell])}^2. \quad (3.46)$$

Finally, the second derivatives are also estimated, in the following way:

$$\begin{aligned} \int_0^\ell (\mathcal{L}f)^2 dx &= \int_0^\ell f_{xx}^2 + G''(v_\ell)^2 f^2 - 2f_{xx}G''(v_\ell)f dx \\ &\geq \int_0^\ell f_{xx}^2 - \frac{1}{2}f_{xx}^2 - 4G''(v_\ell)^2 f^2 dx \\ &\geq \int_0^\ell \frac{1}{2}f_{xx}^2 - 4\left(\sup_{[0, 1]} |G''(y)|\right)^2 f^2 dx. \end{aligned}$$

Rearranging terms and applying (3.46),

$$\begin{aligned} \int_0^\ell f_{xx}^2 dx &\leq 2 \int_0^\ell (\mathcal{L}f)^2 dx + 8\left(\sup_{[0, 1]} |G''(y)|\right)^2 \int_0^\ell f^2 dx \\ &\leq \left(2 + 8\left(\sup_{[0, 1]} |G''(y)|\right)^2 C_3^2\right) \int_0^\ell (\mathcal{L}f)^2 dx. \end{aligned} \quad (3.47)$$

Combining and (3.46) and (3.47) and redefining the constant C_3 appropriately, we arrive at (3.28). \square

3.3 Nonlinear energy-energy-dissipation estimate

3.3.1 Proof of the main proposition

In this subsection, we extend from the linear estimates of Proposition 3.1 to the nonlinear energy-energy-dissipation estimates. We will prove:

Proposition 3.3. *There exist constants $\ell_1, C_1 < \infty$ with the following properties. Let $\ell \geq \ell_1$ and suppose that u satisfies*

$$\begin{aligned} u &\geq 0 & x &\in (0, \ell), \\ u &= 0 & x &\in \{0, \ell\}. \end{aligned} \tag{3.48}$$

Let v be the smooth energy-optimal profile that satisfies (3.29).

Then we have the energy-gap inequality

$$\|u - v\|_{H^1([0, \ell])}^2 \leq C_1(E(u) - E(v)). \tag{3.49}$$

Moreover, if u is “bounded away from $u \equiv 0$ ” in the sense that

$$E(u) \leq \frac{G(0)}{2} \ell_1,$$

then we have the energy-dissipation inequality

$$\|u - v\|_{H^2([0, \ell])}^2 \leq C_1 \int_0^\ell (u_{xx} - G'(u))^2 dx. \tag{3.50}$$

Remark 4. *Since it is easy to bound the energy-gap above by the H^1 -norm of $u - v$, the combination of (3.49) and (3.50) implies*

$$\frac{1}{2C_1} \|u - v\|_{H^1([0, \ell])}^2 \leq E(u) - E(v) \leq \frac{C_1}{2} \int_0^\ell (u_{xx} - G'(u))^2 dx, \tag{3.51}$$

where C_1 has been appropriately redefined. Proposition 1.1 follows by applying (3.51) on each subinterval in-between zeros.

Proof of Proposition 3.3. Let ℓ_3 be as in Proposition 3.1. We use six lemmas, proved in Subsections 3.3.2 and 3.3.3. For the energy-gap inequality, we need the following three lemmas.

Lemma 3.6. *Suppose (3.48) holds and v is defined as usual. For every $\varepsilon > 0$, there exists $\gamma > 0$ such that for all $\ell \geq \ell_3$,*

$$E(u) - E(v) \leq \gamma \quad \Rightarrow \quad \sup_{[0, \ell]} |u - v| \leq \varepsilon.$$

Lemma 3.6 is used to prove the energy gap inequality in the case of small energy gap:

Lemma 3.7. *There exists $\gamma > 0$ such that for all $\ell \geq \ell_3$,*

$$E(u) - E(v) \leq \gamma \quad \Rightarrow \quad \|u - v\|_{H^1([0, \ell])}^2 \leq 4C_3(E(u) - E(v)).$$

In the case of order one energy gap, we use the rough estimate:

Lemma 3.8. *For every $\gamma > 0$, there exists $C_\gamma < \infty$ such that for $u, v \geq 0$,*

$$E(u) - E(v) \geq \gamma \quad \Rightarrow \quad \|u - v\|_{H^1([0,\ell])}^2 \leq C_\gamma(E(u) - E(v)).$$

The combination of Lemmas 3.7 and 3.8 proves (3.49) with

$$C_1 := \max\{4C_3, C_\gamma\}.$$

Similarly, (3.50) is proved by introducing three lemmas. We abbreviate:

$$g := -u_{xx} + G'(u).$$

Lemma 3.9. *Suppose (3.48) holds and v is defined as usual. For every $\varepsilon > 0$, there exists $\gamma > 0$ such that for all $\ell \geq \ell_3$, as long as*

$$E(u) \leq \frac{G(0)}{2} \ell_3, \tag{3.52}$$

then

$$\|g\|_{L^2([0,\ell])}^2 \leq \gamma \quad \Rightarrow \quad \sup_{[0,\ell]} |u - v| \leq \varepsilon. \tag{3.53}$$

Lemma 3.9 is used to prove the energy dissipation inequality in the case of small dissipation:

Lemma 3.10. *There exists $\gamma > 0$ such that for all $\ell \geq \ell_3$, as long as (3.52) holds, then*

$$\|g\|_{L^2([0,\ell])}^2 \leq \gamma \quad \Rightarrow \quad \|u - v\|_{H^2([0,\ell])}^2 \leq 4C_3 \|g\|_{L^2([0,\ell])}^2.$$

In the case of order one dissipation, we use the rough estimate:

Lemma 3.11. *For every $\gamma > 0$, there exists $C_\gamma < \infty$ such that for $u, v \geq 0$ and $E(u) \lesssim 1$,*

$$\|g\|_{L^2([0,\ell])}^2 \geq \gamma \quad \Rightarrow \quad \|u - v\|_{H^2([0,\ell])}^2 \leq C_\gamma \|g\|_{L^2([0,\ell])}^2.$$

The combination of Lemmas 3.10 and 3.11 yields (3.50) with

$$C_1 := \max\{4C_3, C_\gamma\}.$$

□

3.3.2 Proofs of the energy gap lemmas

The main idea in the proof of Lemma 3.6 is to use the function $F : [0, 1] \rightarrow \mathbb{R}$,

$$F(w) := 2 \int_0^w \sqrt{2(G(v) - G(w))} dv + \ell G(w),$$

to prove that for ℓ large,

$$\begin{aligned} E(u) - E(v) &\geq \frac{1}{2} \int_0^{x_m} \left(u_x - \sqrt{2(G(u) - G(M_u))} \right)^2 dx \\ &\quad + \frac{1}{2} \int_{x_m}^\ell \left(u_x + \sqrt{2(G(u) - G(M_u))} \right)^2 dx, \end{aligned}$$

where $M_u := \sup_{[0, \ell]} u$. Roughly, one then concludes that when the energy gap is small, u cannot be far from the function v that solves

$$v_x = \begin{cases} \sqrt{2(G(v) - G(M_v))} & x \in (0, \ell/2), \\ -\sqrt{2(G(v) - G(M_v))} & x \in (\ell/2, \ell). \end{cases} \quad (3.54)$$

Notation 2. As usual, let $f := u - v$.

Proof of Lemma 3.6. We prove the lemma by establishing:

- (1) For every $\varepsilon > 0$, there exist $\ell_* < \infty$ and $\gamma_* > 0$ such that if $\ell \geq \ell_*$ and (3.48) holds, then

$$E(u) - E(v) \leq \gamma_* \quad \Rightarrow \quad \sup_{[0, \ell]} |f| \leq \varepsilon.$$

- (2) For every $\varepsilon > 0$ and $\ell_* < \infty$, there exists $\gamma_{**} > 0$ such that if $\ell \in [\ell_3, \ell_*]$ and (3.48) holds, then

$$E(u) - E(v) \leq \gamma_{**} \quad \Rightarrow \quad \sup_{[0, \ell]} |f| \leq \varepsilon.$$

ad (2): We give an indirect argument. Suppose for a contradiction that there exists a sequence of interval lengths ℓ_n and functions u_n satisfying (3.48) and

$$E_{\ell_n}(u_n) - E_{\ell_n}(v_n) \xrightarrow{n \rightarrow \infty} 0, \quad (3.55)$$

but

$$\lim_{n \rightarrow \infty} \left(\sup_{[0, \ell_n]} |f_n| \right) > 0. \quad (3.56)$$

Without loss of generality, we may assume that $\ell_n \rightarrow \ell \in [\ell_3, \ell_*]$, and that

$$\begin{aligned} u_n &\xrightarrow{H^1} u, \\ u_n &\rightarrow u \quad \text{uniformly,} \end{aligned}$$

where the limit $u \in H^1([0, \ell]) \cap C([0, \ell])$ satisfies (3.48). By (3.55), continuity of $E_\ell(u)$ in ℓ , and lower semi-continuity of $E_\ell(u)$ in u ,

$$E_\ell(u) \leq E_\ell(v_\ell).$$

Hence, by uniqueness of the minimizer among nonnegative functions, $u = v_\ell$. That is, $f = 0$, and by the uniform convergence,

$$\lim_{n \rightarrow \infty} \left(\sup_{[0, \ell_n]} |f_n| \right) = 0,$$

contradicting (3.56).

ad (1): Let x_m be the point at which u achieves its maximum (i.e., $u(x_m) = M_u$), and let $M_v = v(\ell/2)$ be the maximum of v on $[0, \ell]$. We will prove below that for ℓ sufficiently large, we have that

$$\begin{aligned} E(u) - E(v) &\geq \frac{1}{2} \int_0^{x_m} \left(u_x - \sqrt{2(G(u) - G(M_u))} \right)^2 dx \\ &\quad + \frac{1}{2} \int_{x_m}^\ell \left(u_x + \sqrt{2(G(u) - G(M_u))} \right)^2 dx. \end{aligned} \quad (3.57)$$

In view of (3.57), it remains to show that for

$$\begin{aligned} \int_0^{x_m} \left(u_x - \sqrt{2(G(u) - G(M_u))} \right)^2 dx \\ + \int_{x_m}^\ell \left(u_x + \sqrt{2(G(u) - G(M_u))} \right)^2 dx \ll 1, \end{aligned} \quad (3.58)$$

$$E(u) \lesssim 1, \quad (3.59)$$

$$\ell \gg 1, \quad (3.60)$$

then $\sup_{[0, \ell]} |f| \ll 1$. We notice that because $u \geq 0$, (3.59) and (3.60) imply

$$M_u \approx 1 \quad \text{and} \quad \min \{x_m, \ell - x_m\} \gtrsim 1. \quad (3.61)$$

Furthermore, (3.60) implies $M_v \approx 1$, so that in particular,

$$G(M_u) \approx G(M_v). \quad (3.62)$$

From (3.58) and (3.62), we deduce that

$$u_x = \sqrt{2(G(u) - G(M_u))} + r \quad \text{for } x \in (0, x_m),$$

with a small error term r :

$$\int_0^{x_m} r^2 dx \ll 1.$$

Hence, we have by standard ODE theory:

$$\sup_{[0, \min\{x_m, X\}]} |f| \ll 1 \quad \text{for some } X \gg 1. \quad (3.63)$$

We claim that in fact $x_m \gg 1$, so that (3.63) holds on a large neighborhood of zero. To see this, observe that if $x_m \gg 1$ were false, we would have

$$M_u = u(x_m) \stackrel{(3.63)}{\approx} v(x_m) \not\approx 1,$$

contradicting the first part of (3.61). Hence, (3.63) improves to

$$\sup_{[0, X]} |f| \ll 1 \quad \text{for some } X \gg 1.$$

Likewise,

$$\sup_{[\ell - X, \ell]} |f| \ll 1 \quad \text{for some } X \gg 1.$$

Since $v \approx 1$ on $(X, \ell - X)$ and $u \leq 1$, it remains to show that $1 - u \ll 1$ on $(X, \ell - X)$. Suppose that there exists an $x_* \in (X, \ell - X)$ with

$$1 - u(x_*) \not\ll 1. \quad (3.64)$$

Without loss of generality, $x_* \leq x_m$. We compare u to the solution w of

$$\begin{aligned} w_x &= \sqrt{2(G(w) - G(M_u))} & x < x_*, \\ w &= u(x_*) & x = x_*. \end{aligned}$$

Again using (3.58) and standard ODE theory, we deduce

$$\sup_{(x_* - X', x_*)} |u - w| \ll 1 \quad \text{for some } X' \gg 1. \quad (3.65)$$

(Without loss, we may assume that $X' = X$.) On the other hand, because of (3.64), $G(M_u) \ll 1$, and the properties of G , it follows that

$$w \approx -1 \quad \text{on } (-\infty, x_* - X/2). \quad (3.66)$$

From (3.65) and (3.66), we deduce

$$u < 0 \quad \text{on} \quad (x_* - X, x_* - X/2),$$

which contradicts (3.48). This disallows (3.64); hence, we have indeed that $1 - u \ll 1$ on $(X, \ell - X)$.

Thus, it remains only to prove (3.57). To this end, we introduce the function $F : [0, 1] \rightarrow \mathbb{R}$ with

$$F(w) := 2 \int_0^w \sqrt{2(G(v) - G(w))} \, dv + \ell G(w).$$

We will establish the following:

(a) We have the identity

$$\begin{aligned} E(u) &= F(M_u) + \frac{1}{2} \int_0^{x_m} \left(u_x - \sqrt{2(G(u) - G(M_u))} \right)^2 dx \\ &\quad + \frac{1}{2} \int_{x_m}^\ell \left(u_x + \sqrt{2(G(u) - G(M_u))} \right)^2 dx. \end{aligned}$$

(b) For $w \leq M_v$, we have that $E(v) = F(M_v) \leq F(w)$.

(c) The derivative of F is given by

$$F'(w) = G'(w) \left(\ell - \int_0^w \frac{2}{\sqrt{2(G(v) - G(w))}} \, dv \right).$$

(d) For $\ell \gg 1$, we have $F'(M_v) = 0$ and $M_v \approx 1$.

(e) For $\ell \gg 1$, we have $F(M_v) \leq F(w)$ for all $w \in [0, 1]$.

This is enough to conclude (3.57), since then

$$\begin{aligned} E(u) &= \int_0^{x_m} \left(u_x - \sqrt{2(G(u) - G(M_u))} \right)^2 dx \\ &\quad - \int_{x_m}^\ell \left(u_x + \sqrt{2(G(u) - G(M_u))} \right)^2 dx \\ &\stackrel{(a)}{=} F(M_u) \\ &\stackrel{(e)}{\geq} F(M_v) \\ &\stackrel{(b)}{=} E(v). \end{aligned}$$

ad (a): Observe that $G(u) \geq G(M_u)$, so we can express the energy as:

$$\begin{aligned}
E(u) &= \ell G(M_u) + \frac{1}{2} \int_0^{x_m} \left(u_x^2 + \left(\sqrt{2(G(u) - G(M_u))} \right)^2 \right) dx \\
&\quad + \frac{1}{2} \int_{x_m}^{\ell} \left(u_x^2 + \left(\sqrt{2(G(u) - G(M_u))} \right)^2 \right) dx \\
&= \ell G(M_u) + \frac{1}{2} \int_0^{x_m} \left(u_x - \sqrt{2(G(u) - G(M_u))} \right)^2 dx \\
&\quad + \int_0^{x_m} u_x \sqrt{2(G(u) - G(M_u))} dx \\
&\quad + \frac{1}{2} \int_{x_m}^{\ell} \left(u_x + \sqrt{2(G(u) - G(M_u))} \right)^2 dx \\
&\quad - \int_{x_m}^{\ell} u_x \sqrt{2(G(u) - G(M_u))} dx \\
&= F(M_u) + \frac{1}{2} \int_0^{x_m} \left(u_x - \sqrt{2(G(u) - G(M_u))} \right)^2 dx \\
&\quad + \frac{1}{2} \int_{x_m}^{\ell} \left(u_x + \sqrt{2(G(u) - G(M_u))} \right)^2 dx.
\end{aligned}$$

ad (b): The first equality is clear, from (3.54). For the inequality, observe that for $w \leq M_v$ there exists an interval length $\ell_w \leq \ell$ such that the maximum of the solution of (3.29) on ℓ_w is equal to w . Consider the comparison function u that is equal to this solution on $[0, \ell_w/2]$, takes the constant value w on $[\ell_w/2, \ell/2]$, and is even about $\ell/2$. Then

$$\begin{cases} u_x - \sqrt{2(G(u) - G(w))} = 0 & x \in (0, \ell_w/2), \\ u_x + \sqrt{2(G(u) - G(w))} = 0 & x \in (\ell - \ell_w/2, \ell), \\ u_x = \sqrt{2(G(u) - G(w))} = 0 & x \in (\ell_w/2, \ell - \ell_w/2). \end{cases}$$

Consequently,

$$F(w) = E(u) \geq E(v) = F(v),$$

where we have used the minimality of the energy of v .

ad (c): This is a direction calculation.

ad (d): For $x \in [0, \ell/2]$, we have:

$$x = \int_0^{v(x)} \frac{1}{\sqrt{2(G(v) - G(M_v))}} dv,$$

and in particular,

$$\frac{\ell}{2} = \int_0^{M_v} \frac{1}{\sqrt{2(G(v) - G(M_v))}} dv.$$

Thus, (c) implies that $F'(M_v) = 0$. The fact that $M_v \approx 1$ for large ℓ follows, for instance, from the equality in (b) and the definition of F .

ad (e): In view of (b), it suffices to show that $F(M_v) \leq F(w)$ for $w \in [M_v, 1]$ when $\ell \gg 1$. This will follow from $F'(w) \geq 0$ (and the continuity of F) on this interval.

Since $G'(w) \leq 0$ for all $w \in [M_v, 1]$, this means (from (c)) that we would like to show

$$\frac{\ell}{2} \leq \int_0^w \frac{dv}{\sqrt{2(G(v) - G(w))}} \quad \text{for } w \in [M_v, 1].$$

We have equality at M_v (by (d)), therefore it suffices to show

$$\frac{d}{dw} \int_0^w \frac{dv}{\sqrt{G(v) - G(w)}} \geq 0 \quad \text{for } w \in [M_v, 1],$$

(where we have dropped an irrelevant factor of $\sqrt{2}$). For $\ell \gg 1$, $M_v \approx 1$ (by (d)), therefore we would like to show

$$\frac{d}{dw} \int_0^w \frac{dv}{\sqrt{G(v) - G(w)}} \geq 0 \quad \text{for } w \text{ with } 1 - w \ll 1. \quad (3.67)$$

In order to calculate the derivative, we change variables. Let $s := v/w$. Then

$$\begin{aligned} & \frac{d}{dw} \int_0^w \frac{dv}{\sqrt{G(v) - G(w)}} \\ &= \frac{d}{dw} \int_0^1 \frac{w ds}{\sqrt{G(sw) - G(w)}} \\ &= \int_0^1 \frac{ds}{\sqrt{G(sw) - G(w)}} - w \int_0^1 \frac{sG'(sw) - G'(w)}{2(G(sw) - G(w))^{3/2}} ds \\ &= \frac{1}{w} \left(\int_0^w \frac{dv}{\sqrt{G(v) - G(w)}} - \int_0^w \frac{vG'(v) - wG'(w)}{2(G(v) - G(w))^{3/2}} dv \right) \\ &= \frac{1}{w} \left(\int_0^w \frac{G(v) - G(w) + G(v) - vG'(v) - (G(w) - wG'(w))}{2(G(v) - G(w))^{3/2}} dv \right). \quad (3.68) \end{aligned}$$

By hypothesis, $G(v) \geq G(w)$ for $0 \leq v \leq w \leq 1$. We claim that in addition, $w \approx 1$ implies

$$G(v) - v G'(v) - (G(w) - w G'(w)) \geq 0, \quad (3.69)$$

which shows (3.67) and completes the proof of (e). To see this, define

$$h(v) := G(v) - v G'(v).$$

The inequality (3.69) follows from two observations:

- On a neighborhood of $v = 1$, h is monotone decreasing, since $G''(1) > 0$ and

$$h'(v) = -v G''(v).$$

- Away from $v = 1$, h is bounded away from zero, while on the other hand $h(w) \ll 1$ (since $w \approx 1$).

□

Proof of Lemma 3.7. We reexpress the energy difference in a form such that the first term can be bounded using Proposition 3.1, and the second by a Taylor expansion.

$$\begin{aligned} E(u) - E(v) &= \int_0^\ell \left(\frac{1}{2}(u_x^2 - v_x^2) + G(u) - G(v) \right) dx \\ &= \int_0^\ell \left(\frac{1}{2}f_x^2 + v_x f_x + G(u) - G(v) \right) dx \\ &= \int_0^\ell \left(\frac{1}{2}f_x^2 - v_{xx}f + G(u) - G(v) \right) dx \\ &= \int_0^\ell \left(\frac{1}{2}f_x^2 - G'(v)f + G(u) - G(v) \right) dx \\ &= \frac{1}{2} \int_0^\ell (f_x^2 + G''(v)f^2) dx \\ &\quad + \int_0^\ell \left(G(u) - G(v) - G'(v)f - \frac{G''(v)}{2}f^2 \right) dx \\ &\geq \frac{1}{2C_3} \int_0^\ell (f^2 + f_x^2) dx - C_* \sup_{[0,\ell]} |f| \int_0^\ell f^2 dx \end{aligned}$$

with $C_* := \sup_{[0,1]} |G'''(\cdot)|$. By Lemma 3.6, we may choose γ such that

$$\sup_{[0,\ell]} |f| \leq \frac{1}{4C_*C_3},$$

so that

$$E(u) - E(v) \geq \frac{1}{4C_3} \int_0^\ell (f^2 + f_x^2) dx.$$

□

Proof of Lemma 3.8. From the properties of G , there exists a c_* such that for $u \geq 0$,

$$G(u) \geq c_*(u - 1)^2. \quad (3.70)$$

We use the rough estimate:

$$\begin{aligned} \int_0^\ell (f^2 + f_x^2) dx &\leq 2 \int_0^\ell ((u - 1)^2 + (v - 1)^2) dx + 2 \int_0^\ell (u_x^2 + v_x^2) dx \\ &\stackrel{(3.70)}{\leq} \frac{2}{c_*} \int_0^\ell (G(u) + G(v)) dx + 2 \int_0^\ell (u_x^2 + v_x^2) dx \\ &\leq C_{**} (E(u) + E(v)), \end{aligned} \quad (3.71)$$

where $C_{**} := \max\{2/c_*, 4\}$. Letting C_{***} denote the bound on the energy of v for $\ell \in (\ell_3, \infty)$ and adding and subtracting the energy of v in (3.71),

$$\begin{aligned} \int_0^\ell (f^2 + f_x^2) dx &\leq C_{**} (E(u) - E(v)) + 2C_{**} E(v) \\ &\leq C_{**} (E(u) - E(v)) + 2C_{**} C_{***} \\ &\leq C_{**} \left(1 + \frac{2C_{***}}{\gamma} \right) (E(u) - E(v)), \end{aligned}$$

since the energy gap is bounded below by γ . □

3.3.3 Proof of the energy dissipation lemmas

Notation 3. As usual, let $f := u - v$.

Proof of Lemma 3.9. We prove the lemma by establishing:

- (1) For every $\varepsilon > 0$, there exist $\ell_* < \infty$ and $\gamma_* > 0$ such that if $\ell \geq \ell_*$ and u satisfies (3.48) as well as condition (3.52), then

$$\int_0^\ell g^2 dx \leq \gamma_* \quad \Rightarrow \quad \sup_{[0,\ell]} |f| \leq \varepsilon.$$

- (2) For every $\varepsilon > 0$ and $\ell_* < \infty$, there exists $\gamma_{**} > 0$ such that if $\ell \in [\ell_3, \ell_*]$ and u satisfies (3.48) as well as condition (3.52), then

$$\int_0^\ell g^2 dx \leq \gamma_{**} \quad \Rightarrow \quad \sup_{[0, \ell]} |f| \leq \varepsilon.$$

ad (2): As with the analogous statement in the case of the energy gap, we give an indirect argument. Suppose for a contradiction that there exists a sequence of interval lengths ℓ_n and functions u_n satisfying (3.48), (3.52), and

$$\int_0^{\ell_n} g_n^2 dx \xrightarrow{n \rightarrow \infty} 0, \quad (3.72)$$

but

$$\lim_{n \rightarrow \infty} \left(\sup_{[0, \ell_n]} |f_n| \right) > 0. \quad (3.73)$$

Without loss of generality, we may assume that $\ell_n \rightarrow \ell \in [\ell_3, \ell_*]$, and that

$$\begin{aligned} u_n &\xrightarrow{H^2} u, \\ u_n &\rightarrow u \quad \text{uniformly}, \\ g_n &\rightarrow 0 \quad \text{pointwise a.e.} \end{aligned} \quad (3.74)$$

Consequently, we have:

$$u_{xx} - G'(u) = 0 \quad \text{a.e.},$$

and by the uniform convergence, u satisfies (3.48). This implies the dichotomy:

$$\text{either } u \equiv 0 \quad \text{or} \quad u = v.$$

Condition (3.52) survives in the limit $n \rightarrow \infty$ under the convergence (3.74), which rules out the first alternative. Thus, $u = v$, and by the uniform convergence,

$$\lim_{n \rightarrow \infty} \left(\sup_{[0, \ell_n]} |f_n| \right) = 0,$$

contradicting (3.73).

ad (1): We introduce the discrepancy function:

$$\xi := \frac{1}{2}u_x^2 - G(u),$$

so-called because it measures the amount by which u deviates from the equipartition of energy expressed in (3.31). Because

$$\int_0^\ell |\xi| dx \leq E(u) \lesssim 1,$$

we conclude

$$\inf_{[0,\ell]} |\xi| \lesssim \frac{1}{\ell}. \quad (3.75)$$

Moreover, the total variation of ξ is controlled by the energy dissipation:

$$\int_0^\ell |\xi_x| dx = \int_0^\ell |g u_x| dx \leq \left(\int_0^\ell g^2 dx \int_0^\ell u_x^2 dx \right)^{1/2} \lesssim \gamma_*^{1/2}. \quad (3.76)$$

Together, (3.75) and (3.76) imply

$$\sup_{[0,\ell]} |\xi| \lesssim \frac{1}{\ell} + \gamma_*^{1/2}.$$

Hence it is enough to show

$$\sup_{[0,\ell]} |\xi| \ll 1, \quad \ell \gg 1 \quad \Rightarrow \quad \sup_{[0,\ell]} |f| \ll 1.$$

To this end, we use ξ to write u as a perturbed solution of the ODE solved by v :

$$u_x = \pm \sqrt{2(G(u) + \xi)}.$$

Consider $x = 0$. The condition (3.48) implies $u_x(0) \geq 0$, and we have

$$u_x(0) = \sqrt{2(G(u(0)) + \xi(0))} = \sqrt{2(G(0) + \xi(0))} \approx \sqrt{2G(0)},$$

since $G(0)$ is order one and $|\xi| \ll 1$. Therefore, u_x takes the positive square root on a neighborhood of zero:

$$u_x = \sqrt{2(G(u) + \xi)} \quad \text{for } x \in [0, x_m],$$

where x_m denotes the first maximum of u . Notice that $|\xi| \ll 1$ and $u \geq 0$ imply

$$u(x_m) \approx 1. \quad (3.77)$$

Recall now that on $[0, \ell/2]$,

$$v_x = \sqrt{2(G(v) - G(M_v))},$$

and also, $\ell \gg 1$ implies $G(M_v) \ll 1$. Therefore, by ODE theory,

$$\sup_{[0, \min\{x_m, X\}]} |f| \ll 1 \quad \text{for some } X \gg 1. \quad (3.78)$$

Combining (3.77) and (3.78), we see that $x_m \gg 1$ and (3.78) improves to

$$\sup_{[0, X]} |f| \ll 1 \quad \text{for some } X \gg 1.$$

The parallel argument implies

$$\sup_{[\ell-X, \ell]} |f| \ll 1 \quad \text{for some } X \gg 1.$$

Finally, observe that $|\xi| \ll 1$ implies that $\min_{[X, \ell-X]} u \approx 1$. Thus, since also $v \approx 1$ on $(X, \ell - X)$, we have that $|f| \ll 1$ on $(X, \ell - X)$ as well. \square

Proof of Lemma 3.10. Because v satisfies the stationary equation (3.29), we can reexpress the dissipation as:

$$\begin{aligned} \int_0^\ell g^2 dx &= \int_0^\ell (-u_{xx} + G'(u))^2 dx \\ &= \int_0^\ell (-f_{xx} + G'(u) - G'(v))^2 dx \\ &= \int_0^\ell (-f_{xx} + G''(v)f + G'(u) - G'(v) - G''(v)f)^2 dx \\ &\geq \frac{1}{2} \int_0^\ell (-f_{xx} + G''(v)f)^2 dx \\ &\quad - \int_0^\ell (G'(u) - G'(v) - G''(v)f)^2 dx. \end{aligned} \quad (3.79)$$

On the one hand, from Proposition 3.1,

$$\int_0^\ell (-f_{xx} + G''(v)f)^2 dx \geq \frac{1}{C_3} \int_0^\ell (f^2 + f_x^2 + f_{xx}^2) dx.$$

On the other hand,

$$\int_0^\ell (G'(u) - G'(v) - G''(v)f)^2 dx \leq C_*^2 \sup_{[0,\ell]} |f|^2 \int_0^\ell f^2 dx,$$

with $C_* := \sup_{[0,1]} |G'''(\cdot)|$. Thus, (3.79) turns into

$$\int_0^\ell g^2 dx \geq \frac{1}{2C_3} \int (f^2 + f_x^2 + f_{xx}^2) dx - C_*^2 \sup_{[0,\ell]} |f|^2 \int_0^\ell f^2 dx.$$

By Lemma 3.9, we can choose γ so small that

$$\sup_{[0,\ell]} |f| \leq \left(\frac{1}{4C_3 C_*^2} \right)^{1/2},$$

which yields

$$\int_0^\ell g^2 dx \geq \frac{1}{4C_3} \int_0^\ell (f^2 + f_x^2 + f_{xx}^2) dx.$$

□

Proof of Lemma 3.11. It is enough to show

$$\int_0^\ell (f^2 + f_x^2 + f_{xx}^2) dx \lesssim \int_0^\ell g^2 dx + E(u) + 1.$$

Because

$$\int_0^\ell f_x^2 dx = - \int_0^\ell f f_{xx} dx \leq \frac{1}{2} \int_0^\ell (f^2 + f_{xx}^2) dx,$$

it is enough to show

$$\int_0^\ell (f^2 + f_{xx}^2) dx \lesssim \int_0^\ell g^2 dx + E(u) + 1.$$

Because $g = -f_{xx} + G'(u) - G'(v)$,

$$\int_0^\ell f_{xx}^2 dx \lesssim \int_0^\ell g^2 dx + \int_0^\ell f^2 dx,$$

and it is enough to show

$$\int_0^\ell f^2 dx \lesssim E(u) + 1.$$

Finally, since $f^2 \leq 2((u-1)^2 + (v-1)^2)$ and $u, v \geq 0$, we have

$$\int_0^\ell f^2 dx \stackrel{(3.70)}{\lesssim} E(u) + E(v) \lesssim E(u) + 1.$$

□

A Lipschitz constant for the energy

Proof of Lemma 3.1. It suffices to consider the energy in-between a pair of zeros and show that there exists a $C_* < \infty$ such that

$$\left| \frac{d}{d\ell} E(\ell) \right| \leq C_* \exp\left(-\sqrt{G''(1)}\ell\right). \quad (\text{A.1})$$

Let $v : [0, \ell] \rightarrow \mathbb{R}$ be an energy-optimal profile satisfying (3.29). The energy is

$$\begin{aligned} E(\ell) &= \int_0^\ell \left(\frac{1}{2} v_x^2 + G(v) \right) dx \\ &= 2 \int_0^{\ell/2} \left(\frac{1}{2} v_x^2 + G(v) \right) dx \\ &= 2 \int_0^{M_v(\ell)} \frac{\left(\frac{1}{2} v_x^2 + G(v) \right)}{v_x} dv, \end{aligned} \quad (\text{A.2})$$

where $M_v(\ell)$ denotes the maximum of v on $(0, \ell)$. Recalling that

$$\frac{1}{2} v_x^2 = G(v) - G(M_v),$$

we can rewrite (A.2) as

$$\begin{aligned} E(\ell) &= 2 \int_0^{M_v(\ell)} \frac{2G(v) - G(M_v(\ell))}{\sqrt{2(G(v) - G(M_v(\ell)))}} dv \\ &= 2 \int_0^{M_v(\ell)} \sqrt{2(G(v) - G(M_v(\ell)))} dv \\ &\quad + 2G(M_v(\ell)) \int_0^{M_v(\ell)} \frac{1}{\sqrt{2(G(v) - G(M_v(\ell)))}} dv \\ &= 2 \int_0^{M_v(\ell)} \sqrt{2(G(v) - G(M_v(\ell)))} dv + G(M_v(\ell))\ell, \end{aligned} \quad (\text{A.3})$$

where in the last step we have used the identity

$$\begin{aligned} \frac{\ell}{2} &= \int_0^{\ell/2} dx = \int_0^{\ell/2} \frac{v_x}{\sqrt{2(G(v) - G(M_v(\ell)))}} dx \\ &= \int_0^{M_v(\ell)} \frac{1}{\sqrt{2(G(v) - G(M_v(\ell)))}} dv. \end{aligned} \quad (\text{A.4})$$

Differentiation of (A.3) leads to

$$E'(\ell) = G(M_v(\ell)) = \frac{G''(1)}{2}(1 - M_v(\ell))^2 + O((1 - M_v(\ell))^3). \quad (\text{A.5})$$

Thus, it remains to estimate $1 - M_v(\ell)$ in terms of ℓ . To this end, we multiply (A.4) by $\sqrt{G''(M_v(\ell))}$ and then add and subtract the dominant term

$$\int_0^{M_v(\ell)} \frac{dv}{\sqrt{2\lambda(v - M_v(\ell)) + (v - M_v(\ell))^2}} = \int_0^{M_v(\ell)} \frac{dv}{\sqrt{-2\lambda v + v^2}}$$

with

$$\lambda := G'(M_v)/G''(M_v) = -(1 - M_v) + O((1 - M_v)^2). \quad (\text{A.6})$$

This leads to

$$\begin{aligned} & \frac{\sqrt{G''(M_v(\ell))} \ell}{2} \\ &= \int_0^{M_v(\ell)} \left(\frac{\sqrt{G''(M_v(\ell))}}{\sqrt{2(G(v) - G(M_v(\ell)))}} - \frac{1}{\sqrt{-2\lambda v + v^2}} \right) dv \\ & \quad + \int_0^{M_v(\ell)} \frac{dv}{\sqrt{-2\lambda v + v^2}}. \end{aligned} \quad (\text{A.7})$$

The second integral may be calculated analytically:

$$\int_0^{M_v(\ell)} \frac{dv}{\sqrt{-2\lambda v + v^2}} = -\log(-\lambda) + \log\left(M_v(\ell) - \lambda + \sqrt{M_v(\ell)^2 - 2\lambda M_v(\ell)}\right),$$

so that

$$\int_0^{M_v(\ell)} \frac{dv}{\sqrt{-2\lambda v + v^2}} + \log(-\lambda) \stackrel{(\text{A.6})}{=} \log 2 + O(1 - M_v(\ell)). \quad (\text{A.8})$$

Combining (A.6), (A.7), and (A.8), we conclude

$$\begin{aligned} \frac{\sqrt{G''(M_v(\ell))} \ell}{2} + \log(1 - M_v(\ell)) &= \int_0^1 \left(\frac{\sqrt{G''(1)}}{\sqrt{2G(v)}} - \frac{1}{v} \right) dv + \log 2 \\ & \quad + O(1 - M_v(\ell)). \end{aligned}$$

We deduce that to leading order,

$$1 - M_v(\ell) = 2 \exp(-\sqrt{G''(1)}\ell/2) \exp(c),$$

with

$$c := \int_0^1 \left(\frac{\sqrt{G''(1)}}{\sqrt{2G(v)}} - \frac{1}{v} \right) dv.$$

Insertion into (A.5) gives

$$E'(\ell) = 2G''(1) \exp(2c) \exp(-\sqrt{G''(1)} \ell),$$

which verifies (A.1) with $C_* = 2G''(1) \exp(2c)$. \square

Proof of Lemma 3.2. To pass from (3.2) to (3.3) requires only the coarse estimate

$$\begin{aligned} \|v - w\|_{L^2([0,L])} \leq 1 &\Rightarrow \|v - w\|_{L^2([0,L])}^2 + \|v - w\|_{L^2([0,L])} \leq 2, \\ \|v - w\|_{L^2([0,L])} \geq 1 &\Rightarrow \|v - w\|_{L^2([0,L])}^2 + \|v - w\|_{L^2([0,L])} \leq 2\|v - w\|_{L^2([0,L])}^2. \end{aligned}$$

Hence it is enough to show (3.2) and (3.4).

We begin by proving (3.4). Without loss, suppose $x_v \leq x_w \leq y_v \leq y_w$. Let R be given (we will fix it below), and consider separately the intervals

$$(x_v - R, x_w + R), (y_v - R, y_v + R), (x_w + R, y_v - R), (y_w + R, x_v - R).$$

First consider $(x_w + R, y_v - R)$. By fixing R sufficiently large, we have that

$$\min\{G''(v(x)), G''(w(x))\} \geq 1/C_* \quad \text{for } x \in (x_w + R, y_v - R).$$

It follows from Taylor's formula that

$$(v(x) - 1)^2 \leq 2C_* G(v(x)), \quad (w(x) - 1)^2 \leq 2C_* G(w(x)). \quad (\text{A.9})$$

By the boundedness of the energy, it follows that

$$\int_{[0,L]} G(v(x)) dx \lesssim 1, \quad \int_{[0,L]} G(w(x)) dx \lesssim 1. \quad (\text{A.10})$$

Together, (A.9) and (A.10) imply

$$\begin{aligned} &\int_{(x_w+R, y_v-R)} (v - w)^2 dx \\ &\lesssim \int_{(x_w+R, y_v-R)} (v - 1)^2 + (w - 1)^2 dx \lesssim 1. \end{aligned} \quad (\text{A.11})$$

Analogously, we deduce

$$\int_{(y_w+R, x_v-R)} (v-w)^2 dx \lesssim 1. \quad (\text{A.12})$$

Next on $(x_v - R, x_w + R)$ we use the rough estimate

$$|v(x) - w(x)| \leq 2$$

to deduce

$$\int_{(x_v-R, x_w+R)} (v-w)^2 dx \leq 4(x_w + R - (x_v - R)) \lesssim 1 + |x_w - x_v|. \quad (\text{A.13})$$

Similarly, we have that

$$\int_{(y_v-R, y_w+R)} (v-w)^2 dx \lesssim 1 + |y_w - y_v|. \quad (\text{A.14})$$

The combination of (A.11), (A.12), (A.13), and (A.14) gives

$$\int_{[0,L]} (v-w)^2 dx \lesssim 1 + |x_w - x_v| + |y_w - y_v|,$$

from which (3.4) follows.

We now turn to the proof of (3.2). It suffices to establish that there exists $\ell_4 < \infty$ such that for

$$\min\{\ell_v, \ell_w\} \geq \ell_4 \quad \text{and} \quad |x_v - x_w| + |y_v - y_w| \leq \ell_4^{-1} \min\{\ell_v, \ell_w\},$$

the following two statements hold:

- (i) There exists a $\delta_* > 0$ and $C_* < \infty$ such that for

$$|x_v - x_w| + |y_v - y_w| \leq \delta_*,$$

it follows that

$$|x_v - x_w| + |y_v - y_w| \leq C_* \|v - w\|_{L^2([0,L])}.$$

- (ii) For all $\delta > 0$, there exists a constant $C_\delta < \infty$ such that for

$$|x_v - x_w| + |y_v - y_w| \geq \delta,$$

it follows that

$$|x_v - x_w| + |y_v - y_w| \leq C_\delta \|v - w\|_{L^2([0,L])}^2.$$

We begin by proving (ii). Without loss, we may assume

$$|x_v - x_w| \geq |y_v - y_w| \quad \text{and} \quad x_v \leq x_w. \quad (\text{A.15})$$

Notice in particular that

$$x_w \geq x_v + \frac{\delta}{2}. \quad (\text{A.16})$$

It follows that there exists $\tilde{C}_\delta < \infty$ such that for $x \in (x_v + \delta/4, x_w)$, we have

$$v(x) \geq \frac{1}{\tilde{C}_\delta} \quad \text{and} \quad w(x) < 0,$$

so that

$$|v(x) - w(x)| \geq \frac{1}{\tilde{C}_\delta}$$

and

$$\begin{aligned} \|v - w\|_{[0,L]}^2 &\geq \int_{(x_v + \delta/4, x_w)} |v - w|^2 dx \geq \frac{1}{\tilde{C}_\delta^2} \left(x_w - (x_v + \frac{\delta}{2}) \right) \\ &\stackrel{(\text{A.16})}{\geq} \frac{1}{2\tilde{C}_\delta^2} (x_w - x_v) \\ &\stackrel{(\text{A.15})}{\geq} \frac{1}{4\tilde{C}_\delta^2} (|x_v - x_w| + |y_v - y_w|), \end{aligned}$$

which establishes (ii) with $C_\delta = 4\tilde{C}_\delta^2$.

It only remains to prove (i). Without loss, we may again assume (A.15). It is then enough to show that there exists $\delta_* > 0$ and $C_* < \infty$ such that for $|x_v - x_w| \leq \delta_*$, it follows that

$$|x_v - x_w| \leq C_* \left(\int_{(x_v - 1, x_w + 1)} (v - w)^2 dx \right)^{1/2}. \quad (\text{A.17})$$

This will follow from the mean value formula. Suppose to the contrary that there exists a sequence $\{\gamma_n\}_{n=1}^\infty$ with $\gamma_n \rightarrow 0$ and sequences of functions $\{v_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty$ such that

$$0 < x_{w_n} - x_{v_n} \leq \gamma_n \quad (\text{A.18})$$

and

$$|x_{v_n} - x_{w_n}| \geq n \|v_n - w_n\|_{L^2((x_{v_n} - 1, x_{v_n} + 1))}. \quad (\text{A.19})$$

Here v_n satisfies

$$\begin{aligned} -(v_n)_{xx} + G'(v_n) &= 0 & (x_{v_n}, y_{v_n}) \cup (y_{v_n}, x_{v_n}), \\ v_n &> 0 & (x_{v_n}, y_{v_n}), \\ v_n &< 0 & (y_{v_n}, x_{v_n}), \\ v_n &= 0 & \{x_{v_n}, y_{v_n}\}, \end{aligned}$$

and similarly for w_n . Let $L_n^1 := y_{v_n} - x_{v_n}$, $\tilde{L}_n^1 := y_{w_n} - x_{w_n}$, $L_n^2 := x_{v_n} - y_{v_n}$, and $\tilde{L}_n^2 := x_{w_n} - y_{w_n}$. Without loss we may assume that

$$\begin{aligned} L_n^1 &\rightarrow L^1 \in [\ell_4, \infty), \\ L_n^2 &\rightarrow L^2 \in [\ell_4, \infty), \\ \tilde{L}_n^1 &\rightarrow \tilde{L}^1 \in [\ell_4, \infty), \\ \tilde{L}_n^2 &\rightarrow \tilde{L}^2 \in [\ell_4, \infty). \end{aligned}$$

Shifting if necessary, we may also assume that

$$x_{v_n} = 0 \quad \forall n \in \mathbb{N} \tag{A.20}$$

From (A.18), we deduce that

$$x_{w_n} \rightarrow 0.$$

From (A.19), we deduce that

$$\|v_n - w_n\|_{L^2((-1,1))} \rightarrow 0,$$

from which it follows that

$$L^1 = \tilde{L}^1, \quad L^2 = \tilde{L}^2.$$

By now estimating the rate of convergence, we will derive a contradiction. We have

$$\begin{aligned} |x_{w_n}| &\stackrel{(A.20)}{=} |x_{v_n} - x_{w_n}| \stackrel{(A.19)}{\geq} n \|v_n - w_n\|_{L^2((-1,1))} \\ &\geq n \left(\|v_n(\cdot) - v_n(\cdot + x_{w_n})\|_{L^2((-1,1))} \right. \\ &\quad \left. - \|v_n(\cdot + x_{w_n}) - w_n(\cdot)\|_{L^2((-1,1))} \right), \end{aligned} \tag{A.21}$$

by the triangle inequality. From the mean value theorem, for γ_n sufficiently small, it follows that the first term on the right-hand side is estimated by

$$\|v_n(\cdot) - v_n(\cdot + x_{w_n})\|_{L^2((-1,1))} \geq |x_{w_n}| \frac{\|(v_n)_x\|_{L^2((-1,1))}}{2}. \tag{A.22}$$

For the second term on the right-hand side of (A.21), we will use an ODE argument. Consider the interval $(x_{w_n}, 1)$. Notice that $v_n(\cdot + x_{w_n})$ and $w_n(\cdot)$ satisfy the same Dirichlet condition at $x = 0$ and the ODEs

$$\begin{aligned}\frac{d}{dx}v_n(x + x_{w_n}) &= \sqrt{G(v_n(x + x_{w_n})) - G(M_{v_n})}, \\ \frac{d}{dx}w_n(x) &= \sqrt{G(w_n(x)) - G(M_{w_n})},\end{aligned}$$

where $M_{v_n} = \max_{x \in (0, y_{v_n})} v_n(x)$, and similarly for w_n . Notice that for L sufficiently large, there exists $\varepsilon > 0$ (independent of n) such that

$$\min\{M_{v_n}, M_{w_n}\} \geq 1 - \varepsilon/2 \quad (\text{A.23})$$

while at the same time

$$\sup_{x \in (x_{w_n}, 1)} \max\{v_n(x + x_{w_n}), w_n(x)\} \leq 1 - \varepsilon. \quad (\text{A.24})$$

By continuous dependence of the solutions of ODEs on parameters, we have that

$$\begin{aligned}& \sup_{x \in (-1, 1)} |v_n(x + x_{w_n}) - w_n(x)| \\ & \leq \sup_{\lambda \in [1 - \varepsilon/2, 1]} \sup_{s \in [0, 1 - \varepsilon]} \frac{d}{d\lambda} \left| \sqrt{G(s) - G(\lambda)} \right| |M_{v_n} - M_{w_n}|.\end{aligned} \quad (\text{A.25})$$

By (A.23) and (A.24), the first term on the right-hand side of (A.25) is uniformly bounded. On the other hand, for the second term we have

$$\begin{aligned}|M_{v_n} - M_{w_n}| &\leq |\exp(-cL_n) - \exp(-c\tilde{L}_n)| \\ &\leq \exp(-c \min\{L_n, \tilde{L}_n\}) |L_n - \tilde{L}_n|.\end{aligned} \quad (\text{A.26})$$

Finally, notice that

$$\begin{aligned}|L_n - \tilde{L}_n| &\leq |x_{v_n} - x_{w_n}| + |y_{v_n} - y_{w_n}| \\ &\stackrel{(\text{A.15})}{\leq} 2|x_{v_n} - x_{w_n}| = 2|x_{w_n}|,\end{aligned} \quad (\text{A.27})$$

so that (A.26) becomes

$$|M_{v_n} - M_{w_n}| \leq 2|x_{w_n}| \exp(-c \min\{L_n, \tilde{L}_n\}). \quad (\text{A.28})$$

Combining (A.21), (A.22), (A.28), and (A.25), we arrive at

$$|x_{w_n}| \geq n \left(|x_{w_n}| \frac{\|(v_n)_x\|_{L^2((-1, 1))}}{2} - C \exp(-c \min\{L_n, \tilde{L}_n\}) |x_{w_n}| \right).$$

Dividing through by $|x_{w_n}|$ gives

$$1 \geq n \left(\frac{\|(v_n)_x\|_{L^2((-1,1))}}{2} - C \exp(-c \min\{L_n, \tilde{L}_n\}) \right). \quad (\text{A.29})$$

As long as ℓ_4 is at least order one, we have that $\|(v_n)_x\|_{L^2((-1,1))}$ is bounded away from zero, while by choosing ℓ_4 sufficiently large we have that

$$C \exp(-c \min\{L_n, \tilde{L}_n\}) \ll 1.$$

Hence we conclude from (A.29) that for some constant $c > 0$,

$$1 \geq n c,$$

which for $n \rightarrow \infty$ gives a contradiction. □

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References

- [A] S. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math. **390** (1988), 79-96.
- [BK] L. Bronsard and R. V. Kohn, *On the slowness of phase boundary motion in one space dimension*, Comm. Pure App. Math. **43** (1990), 983-997.
- [CP] J. Carr and R. L. Pego, *Metastable patterns in solutions of $u_t = \epsilon^2 u_{xx} - f(u)$* , Comm. Pure App. Math. **42** (1989), 523-576.
- [C] X. Chen, *Generation, propagation, and annihilation of metastable patterns*, J. Differential Equations **206** (2004), 399-437.
- [ER] J. P. Eckmann and J. Rougemont, *Coarsening by Ginzburg-Landau dynamics*, Comm. Math. Phys. **199** (1998), 441-470.

- [FM] P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to traveling front solutions*, Arch. Rational Mech. Anal. **65** (1977), 335-361.
- [FH] G. Fusco and J. K. Hale, *Slow-motion manifolds, dormant instability, and singular perturbations*, J. Dynam. Differential Equations **1** (1989), 75-94.
- [G] C. P. Grant, *Slow motion in one-dimensional Cahn-Morral systems*, SIAM J. Math. Anal. **26** (1995), 21-34.
- [MS] P. de Mottoni and M. Schatzman, *Development of interfaces in \mathbb{R}^N* , Proc. Roy. Soc. Edinburgh Sect. A **116** (1990), 207-220.
- [R] J. Rougemont, *Dynamics of kinks in the Ginzburg-Landau equation: approach to a metastable shape and collapse of embedded pairs of kinks*, Nonlinearity **12** (1999), 539-554.
- [W] M. J. Ward, *Metastable patterns, layer collapses, and coarsening for a one dimensional Ginzburg-Landau equation*, Studies in Appl. Math. **91** (1994), 51-93.