1. Let $A$ be an integral domain, and let $S$ be a multiplicative subset of $A$.
   (a) Prove that every ideal of $S^{-1}A$ is of the form $S^{-1}I$ for some ideal $I$ of $A$.
   (b) Let $g : A \rightarrow B$ be a ring homomorphism such that $g(s)$ is a unit in $B$ for all $s \in S$. Prove that there is a unique ring homomorphism $h : S^{-1}A \rightarrow B$ such that $g = h \circ f$, where $f : A \rightarrow S^{-1}A$ is the natural inclusion.

2. Let $R$ be a Dedekind ring, and let $S$ be a finite subset of nonzero prime ideals of $R$.
   (a) Prove that $R^* = \cap_p R_p^*$.
   (b) Show that there is a canonical exact sequence of abelian groups
   $$1 \rightarrow R^* \rightarrow (R^S)^* \rightarrow \oplus_{p \in S} \left(K^*/R_p^*\right) \rightarrow \text{Cl}(R) \rightarrow \text{Cl}(R^S) \rightarrow 1.$$ 
   (c) Prove that $K^*/R_p^* \cong \mathbb{Z}$ for each $p \in S$.
   (d) If $K$ is a number field and $R = \mathcal{O}_K$, use Dirichlet’s unit theorem to show that
   $$(R^S)^* \cong W_K \times \mathbb{Z}^{r_1+r_2-1+|S|}.$$ 

3. Let $\overline{\mathbb{Q}}$ denote an algebraic closure of $\mathbb{Q}$. Show that a subgroup $G$ of $\overline{\mathbb{Q}}^*$ is finitely generated if and only if $G \subseteq (\mathcal{O}_K^S)^*$ for some number field $K$ and some finite set $S$ of nonzero prime ideals of $\mathcal{O}_K$.

4. Prove that $e$ and $f$ are multiplicative in towers, in the sense that if $p_1 \subset p_2 \subset p_3$ are nonzero prime ideals contained in the number rings $A_1 \subset A_2 \subset A_3$, then $e(p_3/p_1) = e(p_3/p_2) \cdot e(p_2/p_1)$ and $f(p_3/p_1) = f(p_3/p_2) \cdot f(p_2/p_1)$. 

5. Find a prime number $p$ and quadratic extensions $K$ and $L$ of $\mathbb{Q}$ illustrating of each of the following:

(a) $p$ can be totally ramified in $K$ and $L$ without being totally ramified in $KL$.

(b) $K$ and $L$ can each contain unique primes lying over $p$ while $KL$ does not.

(c) $p$ can be inert in $K$ and $L$ without being inert in $KL$.

(d) The residue degrees of $p$ in $K$ and $L$ can be 1 without being 1 in $KL$.

6. Let $L/K$ be a Galois extension of number fields. Suppose $q$ is a nonzero prime ideal of $\mathcal{O}_L$ lying over the nonzero prime ideal $p$ of $\mathcal{O}_K$. Let $D = D_{q/p}$ and $I = I_{q/p}$. Show that:

(a) $L^D$ is the smallest intermediate field $K'$ such that $q$ is the only prime ideal of $\mathcal{O}_L$ lying over $p' = \mathcal{O}_{K'} \cap q$.

(b) $L^I$ is the smallest intermediate field $K'$ such that $q$ is totally ramified over $p' = \mathcal{O}_{K'} \cap q$. 