# Homework Assignment # 7

Due Wednesday, November 22

1. Let $A$ be an integral domain, and let $S$ be a multiplicative subset of $A$.
   (a) Prove that every ideal of $S^{-1}A$ is of the form $S^{-1}I$ for some ideal $I$ of $A$.
   (b) Let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in $B$ for all $s \in S$. Prove that there is a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$, where $f : A \to S^{-1}A$ is the natural inclusion.
2. Let $R$ be a Dedekind ring, and let $S$ be a finite subset of nonzero prime ideals of $R$.
   (a) Prove that $R^* = \cap_{\mathfrak{p}} R^*_{\mathfrak{p}}$.
   (b) Show that there is a canonical exact sequence of abelian groups
   $$1 \to R^* \to (R^S)^* \to \oplus_{\mathfrak{p}\in S}\left(K^*/R^*_{\mathfrak{p}}\right) \to \mathrm{Cl}(R) \to \mathrm{Cl}(R^S) \to 1\ .$$
   (c) Prove that $K^*/R^*_{\mathfrak{p}} \cong \mathbf{Z}$ for each $\mathfrak{p} \in S$.
   (d) If $K$ is a number field and $R = \mathcal{O}_K$, use Dirichlet's unit theorem to show that
   $$(R^S)^* \cong W_K \times \mathbf{Z}^{r_1+r_2-1+|S|}\ .$$
3. Let $\overline{\mathbf{Q}}$ denote an algebraic closure of $\mathbf{Q}$. Show that a subgroup $G$ of $\overline{\mathbf{Q}}^*$ is finitely generated if and only if $G \subseteq (\mathcal{O}_K^S)^*$ for some number field $K$ and some finite set $S$ of nonzero prime ideals of $\mathcal{O}_K$.
4. Prove that $e$ and $f$ are multiplicative in towers, in the sense that if $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \mathfrak{p}_3$ are nonzero prime ideals contained in the number rings $A_1 \subset A_2 \subset A_3$, then $e(\mathfrak{p}_3/\mathfrak{p}_1) = e(\mathfrak{p}_3/\mathfrak{p}_2)\cdot e(\mathfrak{p}_2/\mathfrak{p}_1)$ and $f(\mathfrak{p}_3/\mathfrak{p}_1) = f(\mathfrak{p}_3/\mathfrak{p}_2)\cdot f(\mathfrak{p}_2/\mathfrak{p}_1)$.

5. Find a prime number $p$ and quadratic extensions $K$ and $L$ of $\mathbf{Q}$ illustrating of each of the following:

   (a) $p$ can be totally ramified in $K$ and $L$ without being totally ramified in $KL$.

   (b) $K$ and $L$ can each contain unique primes lying over $p$ while $KL$ does not.

   (c) $p$ can be inert in $K$ and $L$ without being inert in $KL$.

   (d) The residue degrees of $p$ in $K$ and $L$ can be 1 without being 1 in $KL$.

6. Let $L/K$ be a Galois extension of number fields. Suppose $\mathfrak{q}$ is a nonzero prime ideal of $\mathcal{O}_L$ lying over the nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$. Let $D = D_{\mathfrak{q}/\mathfrak{p}}$ and $I = I_{\mathfrak{q}/\mathfrak{p}}$. Show that:

   (a) $L^D$ is the smallest intermediate field $K'$ such that $\mathfrak{q}$ is the only prime ideal of $\mathcal{O}_L$ lying over $\mathfrak{p}' = \mathcal{O}_{K'} \cap \mathfrak{q}$.

   (b) $L^I$ is the smallest intermediate field $K'$ such that $\mathfrak{q}$ is totally ramified over $\mathfrak{p}' = \mathcal{O}_{K'} \cap \mathfrak{q}$.