Potential Theory on Berkovich Spaces
Lecture 1: The Berkovich Projective Line

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Arizona Winter School on p-adic Geometry
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- In particular, we will introduce potential theory (Laplacians, harmonic and subharmonic functions, . . .) on such spaces. The results obtained closely parallel classical facts over $\mathbb{C}$ (e.g. the maximum modulus principle, Poisson formula. . .)
- The ultimate goal (which we will unfortunately not say much about) is to treat archimedean and non-archimedean analytic spaces in a unified way, and to make precise Arakelov's analogy between intersection theory on arithmetic surfaces and potential theory on Riemann surfaces.

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- The Berkovich affine line $\mathbb{A}^1_{\mathrm{Berk}}$ over K is a locally compact, Hausdorff, and uniquely path-connected topological space which contains K as a dense subspace. The Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk}}$ is obtained by adjoining a point ∞ to $\mathbb{A}^1_{\mathrm{Berk}}$.

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- One can view $\mathbb{P}^1_{\mathrm{Berk}}$ as a profinite $\mathbb{R}$-tree. This allows one to define a Laplacian operator on $\mathbb{P}^1_{\mathrm{Berk}}$ which comes from the usual Laplacian on a finite graph. The tree structure also leads to a good theory of harmonic and subharmonic functions which closely parallels the classical theory over $\mathbb{C}$.

Multiplicative seminorms

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As a set, $\mathbb{A}^1_{\mathrm{Berk},K}$ consists of all multiplicative seminorms on the polynomial ring $K[T]$ which extend the usual absolute value on K.

Multiplicative seminorms (continued)

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1. We will assume throughout that our field K is complete and algebraically closed.
2. We will identify seminorms $|\ |_x$ with points $x \in \mathbb{A}^1_{\mathrm{Berk},K}$.
3. We will usually omit explicit reference to the ground field K, writing $\mathbb{A}^1_{\mathrm{Berk}}$.

Definition

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Explicitly, a fundamental system of open neighborhoods is given by open sets of the form

$$\{x \in \mathbb{A}^1_{\mathrm{Berk}} \ : \ \alpha_i < |f_i|_x < \beta_i\}$$

with $f_1, \ldots, f_m \in K[T]$ and $\alpha_i, \beta_i \in \mathbb{R}$ $(i = 1, \ldots, m)$.

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Example

Fix a closed disk $B(a,r) = \{z \in K \ : \ |z-a| \leq r\}$ in K, and define $| \ |_{B(a,r)}$ by

$$|f|_{B(a,r)} = \sup_{z \in B(a,r)} |f(z)|.$$

Then $| \ |_{B(a,r)}$ is a multiplicative seminorm on $K[T]$ (by Gauss' lemma).

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- In particular, K embeds into $\mathbb{A}^1_{\mathrm{Berk}}$ as the set of disks of radius zero, and is dense in the Berkovich topology.
- Similarly, $\mathbb{P}^1(K)$ can naturally be embedded as a dense subset of $\mathbb{P}^1_{\mathrm{Berk}}$.

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- In this way we have “connected up” the totally disconnected space K by adding points corresponding to closed disks!

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- We need to add points corresponding to such sequences in order to obtain a compact space, since if $\{B(a_n, r_n)\}$ is any decreasing nested sequence of closed disks, the map

  $$f \mapsto \lim_{n\to\infty} |f|_{B(a_n,r_n)}$$

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- Two such sequences of disks with empty intersection define the same seminorm if and only if the sequences are cofinal.

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Theorem (Berkovich's Classification Theorem)

Every point $x \in \mathbb{A}^1_{\mathrm{Berk}}$ corresponds to a nested sequence $B(a_1, r_1) \supseteq B(a_2, r_2) \supseteq B(a_3, r_3) \supseteq \cdots$ of closed disks, in the sense that

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Type IV: $B = \emptyset$.

The Gauss norm

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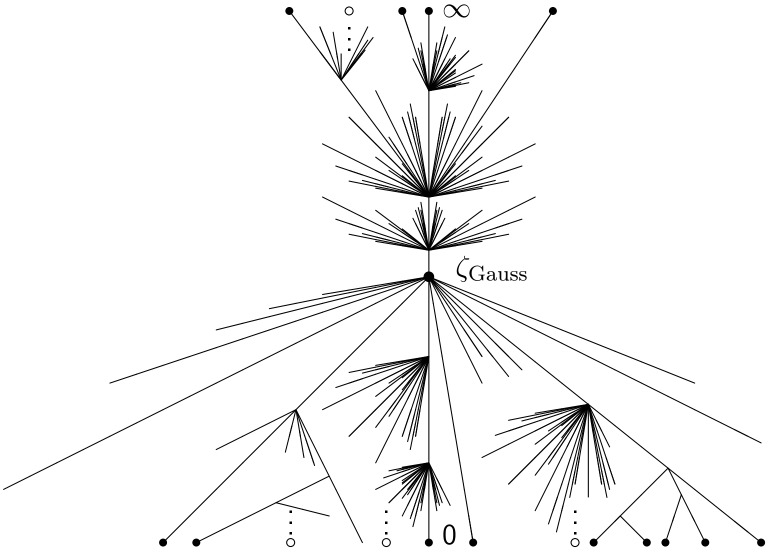
- Following the terminology introduced by Chambert-Loir, the distinguished point $\zeta_{\mathrm{Gauss}} = \zeta_{0,1}$ in $\mathbb{A}^1_{\mathrm{Berk}}$ corresponding to the Gauss norm

$$|\sum_{i=0}^{n} a_i T^i|_{\mathrm{Gauss}} = \max |a_i|$$

on $K[T]$ will be called the Gauss point.

A visual representation of $\mathbb{P}^1_{\mathrm{Berk}}$



Alternate representation of $\mathbb{P}^1_{\mathrm{Berk}}$

Visualizing $\mathbb{P}^1_{\text{Berk}}$

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- Some of the branches extend all the way to the bottom (terminating in points of type I), while others are "cauterized off" earlier and terminate at points of type IV. In any case, every branch terminates either at a point of type I or type IV.

Tangent directions

Definition

Let $x \in \mathbb{P}^1_{\mathrm{Berk}}$. The space T_x of tangent directions at x is the set of equivalence classes of paths $\ell_{x,y}$ emanating from x, where y is any point of $\mathbb{P}^1_{\mathrm{Berk}}$ not equal to x. Two paths $\ell_{x,y_1}, \ell_{x,y_2}$ are equivalent if they share a common initial segment.

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- We denote by $U(x; \vec{v})$ the connected component of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{x\}$ corresponding to $\vec{v} \in T_x$.

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- We denote by $U(x; \vec{v})$ the connected component of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{x\}$ corresponding to $\vec{v} \in T_x$.
- The open sets $U(x; \vec{v})$ for $x \in \mathbb{P}^1_{\mathrm{Berk}}$ and $\vec{v} \in T_x$ generate the topology on $\mathbb{P}^1_{\mathrm{Berk}}$.

Berkovich disks

For $a \in K$ and $r > 0$, write

$$\mathcal{B}(a,r)^- = \{x \in \mathbb{A}^1_{\mathrm{Berk}} : |T-a|_x < r\},$$
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- We call a set of the form $\mathcal{B}(a,r)^-$ an open Berkovich disk in $\mathbb{A}^1_{\mathrm{Berk}}$, and a set of the form $\mathcal{B}(a,r)$ a closed Berkovich disk in $\mathbb{A}^1_{\mathrm{Berk}}$.

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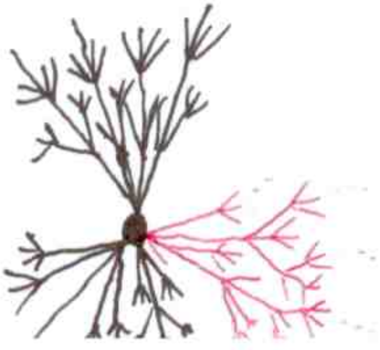
- Similarly, we can define open and closed Berkovich disks in $\mathbb{P}^1_{\mathrm{Berk}}$.
- The intersection of a Berkovich open disk with $\mathbb{P}^1(K)$ is a (classical) open disk (and similarly for closed disks).

A Berkovich open disk



Lemma

Every open set $U(x;\vec{v})$ with x of type II or III and $\vec{v} \in T_x$ is a Berkovich open disk, and conversely.

Simple domains

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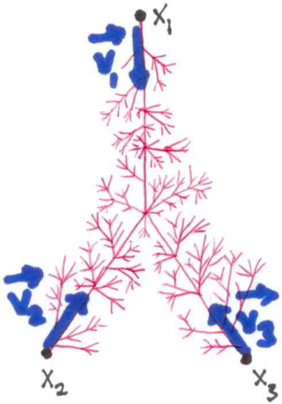
Finite intersections of Berkovich open disks in $\mathbb{P}^1_{\mathrm{Berk}}$ are called simple domains, and they form a fundamental system of open neighborhoods for the topology on $\mathbb{P}^1_{\mathrm{Berk}}$.

Ends of a simple domain

A simple domain V in $\mathbb{P}^1_{\mathrm{Berk}}$ has a finite set $x_1, \ldots, x_n$ of boundary points, and a corresponding finite set $\vec{v}_1, \ldots, \vec{v}_n$ of ends, which are the inward-pointing tangent directions:

Ends of a simple domain

A simple domain V in $\mathbb{P}^1_{\mathrm{Berk}}$ has a finite set $x_1, \ldots, x_n$ of boundary points, and a corresponding finite set $\vec{v}_1, \ldots, \vec{v}_n$ of ends, which are the inward-pointing tangent directions:



Tangent directions at ζ_{Gauss}

- The tangent directions $\vec{v} \in T_{\zeta_{\text{Gauss}}}$ correspond bijectively to elements of $\mathbb{P}^1(\tilde{K})$, the projective line over the residue field of K.

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- The correspondence between elements of $T_{\zeta_{\text{Gauss}}}$ and open disks is given explicitly by $\vec{v} \mapsto U(\zeta_{\text{Gauss}}; \vec{v})$.

Tangent directions at a general point

- More generally, for each point $x = \zeta_{a,r}$ of type II, the set T_x of tangent directions at x is (non-canonically) isomorphic to $\mathbb{P}^1(\tilde{K})$: there is one tangent direction going "up" to infinity, and the other tangent directions correspond to open disks $B(a', r)^-$ of radius r contained in $B(a, r)$.

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- For $x \in \mathbb{P}^1_{\mathrm{Berk}}$, we have:

$$|T_x| = \begin{cases} |\mathbb{P}^1(\tilde{K})| & x \text{ of type II} \\ 2 & x \text{ of type III} \\ 1 & x \text{ of type I or type IV.} \end{cases}$$

Following notation introduced by Juan Rivera-Letelier, we write $\mathbf{H}_{\mathrm{Berk}}$ for the subset of $\mathbb{P}^1_{\mathrm{Berk}}$ consisting of all points of type II, III, or IV.

- We refer to $\mathbf{H}_{\mathrm{Berk}}$ as "Berkovich hyperbolic space".

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- We write $\mathbf{H}^{\mathbb{Q}}_{\mathrm{Berk}}$ for the set of type II points, and $\mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$ for the set of points of type II or III.
- The subset $\mathbf{H}^{\mathbb{Q}}_{\mathrm{Berk}}$ is dense in $\mathbb{P}^1_{\mathrm{Berk}}$.

The diameter function on $\mathbb{A}^1_{\mathrm{Berk}}$

Define the diameter function $\mathrm{diam} : \mathbb{A}^1_{\mathrm{Berk}} \to \mathbb{R}_{\geq 0}$ by setting $\mathrm{diam}(x) = \lim r_i$ if x corresponds to the nested sequence $\{B(a_i, r_i)\}$.

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- Because K is complete, if x is of type IV, then necessarily $\mathrm{diam}(x) > 0$. Thus $\mathrm{diam}(x) = 0$ for $x \in \mathbb{A}^1_{\mathrm{Berk}}$ of type I, and $\mathrm{diam}(x) > 0$ for $x \in \mathbf{H}_{\mathrm{Berk}}$.

A partial order on $\mathbb{A}^1_{\mathrm{Berk}}$

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- For each pair of points $x, y \in \mathbb{A}^1_{\mathrm{Berk}}$, there is a unique least upper bound $x \vee y$ in $\mathbb{A}^1_{\mathrm{Berk}}$ with respect to this partial order.
- Concretely, if $x = \zeta_{a,r}$ and $y = \zeta_{b,s}$ are points of type I, II or III, then $x \vee y$ is the point of $\mathbb{A}^1_{\mathrm{Berk}}$ corresponding to the smallest disk containing both $B(a, r)$ and $B(b, s)$.

The path metric on $\mathbf{H}_{\mathrm{Berk}}$

- If $x, y \in \mathbf{H}_{\mathrm{Berk}}$ with $x \le y$, define the path metric

$$\rho(x, y) = \log_v \frac{\mathrm{diam}(y)}{\mathrm{diam}(x)} \ ,$$

where $\log_v$ denotes the logarithm to the base q_v, with $q_v > 1$ a suitable constant.

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- For example, if $K = \mathbb{C}_p$ and $|p|_p = 1/p$, we would set $q_v = p$ in order to have $\{\log_v |x|_p \ : \ x \in \mathbb{C}_p^*\} = \mathbb{Q}$.

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- More generally, for $x, y \in \mathbf{H}_{\mathrm{Berk}}$ arbitrary, we define

$$\rho(x,y) = \rho(x, x \vee y) + \rho(y, x \vee y) \ .$$

  This gives an metric on $\mathbf{H}_{\mathrm{Berk}}$.

Remarks on the path metric on $\mathbf{H}_{\mathrm{Berk}}$

- For closed disks $B(a,r) \subseteq B(a,R)$, we have
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- The topology on $\mathbf{H}_{\mathrm{Berk}}$ defined by the metric ρ is not the subspace topology induced from the Berkovich topology on $\mathbb{P}^1_{\mathrm{Berk}}$. However, the inclusion map $\mathbf{H}_{\mathrm{Berk}} \hookrightarrow \mathbb{P}^1_{\mathrm{Berk}}$ is continuous with respect to these topologies.

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- The topology on $\mathbf{H}_{\text{Berk}}$ defined by the metric ρ is not the subspace topology induced from the Berkovich topology on $\mathbb{P}^1_{\text{Berk}}$. However, the inclusion map $\mathbf{H}_{\text{Berk}} \hookrightarrow \mathbb{P}^1_{\text{Berk}}$ is continuous with respect to these topologies.
- The group $\text{PGL}(2,K)$ of Möbius transformations acts continuously on $\mathbb{P}^1_{\text{Berk}}$ in a natural way compatible with the usual action on $\mathbb{P}^1(K)$, and this action preserves $\mathbf{H}_{\text{Berk}}$. One can show that $\text{PGL}(2,K)$ acts via isometries on $\mathbf{H}_{\text{Berk}}$, i.e.,
$$\rho(M(x), M(y)) = \rho(x,y)$$
for all $x, y \in \mathbf{H}_{\text{Berk}}$ and all $M \in \text{PGL}(2,K)$. (This shows that the metric ρ is canonical).

The canonical distance on $\mathbb{A}^1_{\mathrm{Berk}}$

- The diameter function can be used to extend the usual distance function $|x - y|$ on K to $\mathbb{A}^1_{\mathrm{Berk}}$ in a natural way by setting

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- We call this extension the canonical distance (or Hsia kernel) on $\mathbb{A}^1_{\mathrm{Berk}}$ (relative to infinity).
- If $x, y \in K$, then $[x, y]_\infty = |x - y|$.

More generally:

If $x = \zeta_{a,r}$ and $y = \zeta_{b,s}$ are points of Type I,II, or III, then

$$[x, y]_\infty = \sup_{\substack{x_0 \in B(a,r) \\ y_0 \in B(b,s)}} |x_0 - y_0|.$$

Properties of $[x, y]_\infty$

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3. For all $x, y, z \in \mathbb{A}^1_{\mathrm{Berk}}$, we have the ultrametric inequality

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   with equality if $[x, z]_\infty \neq [y, z]_\infty$.
4. $[x, y]_\infty$ satisfies all of the axioms for an ultrametric except we have $[x, x]_\infty > 0$ for $x \in \mathbf{H}_{\mathrm{Berk}}$.