Potential Theory on Berkovich Spaces
Lecture 3: Harmonic functions

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Goals

In this lecture, we will explore the notion of a harmonic function in the context of $\mathcal{M}(\mathbb{Z})$ and $\mathbb{P}^1_{\mathrm{Berk}}$.

Notation and terminology

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- A domain in a topological space X is a connected open subset of X.

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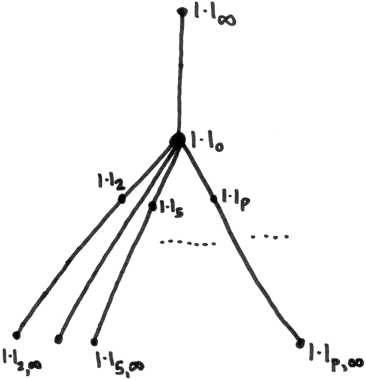
$\zeta_{p,\infty}$: the point of $\mathcal{M}(\mathbb{Z})$ corresponding to the p-trivial seminorm $|\ |_{p,\infty}$.

A picture of $\mathcal{M}(\mathbb{Z})$



Tangent directions in $\mathcal{M}(\mathbb{Z})$

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- At all other points of $\mathcal{M}(\mathbb{Z})$, the space T_x has cardinality 1 or 2.
- For $v \in M_{\mathbb{Q}}$, we will refer to the segment

$$\ell_v = \begin{cases} \{|\ |_{\infty,\epsilon}\}_{0\leq\epsilon\leq 1} & v \leftrightarrow \infty \\ \{|\ |_{p,\epsilon}\}_{0\leq\epsilon\leq\infty} & v \leftrightarrow p. \end{cases}$$

as the branch emanating from ζ_0 in the direction v.

The metric space $\mathbf{H}_{\mathbb{Z}}$

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- The points of $\mathbf{H}_{\mathbb{Z}}$ are precisely the ones at finite distance from the trivial point ζ_0.
- The space $\mathbf{H}_{\mathbb{Z}}$ is endowed with a metric topology which is not the same as its subspace topology as a subset of $\mathcal{M}(\mathbb{Z})$.

Affine functions

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  2. f is constant (i.e, $a_v = 0$) on all but finitely many branches ℓ_v.

Harmonic functions

If f is an affine function on $\mathcal{M}(\mathbb{Z})$, define

$$\Delta_{\zeta_0}(f) = -\sum_{v \in T_{\zeta_0}} d_{\vec{v}} f(x)$$
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Motivated by the classical notion of harmonic functions on a weighted graph, we make the following definition:

Definition

An affine function $f : \mathcal{M}(\mathbb{Z}) \to \mathbb{R} \cup \{\pm\infty\}$ is harmonic at ζ_0 if $\Delta_{\zeta_0}(f) = 0$.

Example: $\log |n|$

Lemma

If $n \in \mathbb{Z}$ is a nonzero integer, then the function $x \mapsto -\log |n|_x$ is affine, and is harmonic at ζ_0.

Proof.

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- In particular, $-\log|n|_x$ is constant along ℓ_v for all finite places v corresponding to a prime p with $p \nmid n$.
- The fact that $-\log|n|_x$ is harmonic at ζ_0 is equivalent to the product formula for $\mathbb{Q}$:

$$\Delta_{\zeta_0}(-\log|n|_x) = \sum_{v \in M_{\mathbb{Q}}} \log|n|_v = 0.$$

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Harmonic functions on $\mathbb{P}^1_{\mathrm{Berk}}$

- We now turn to what it means for an (extended) real-valued function on $\mathbb{P}^1_{\mathrm{Berk}}$ to be harmonic. This is more complicated than the corresponding notion for $\mathcal{M}(\mathbb{Z})$, since the branching behavior of $\mathbb{P}^1_{\mathrm{Berk}}$ is much wilder.

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- Recall that if $x \in \mathbb{P}^1_{\mathrm{Berk}}$, the set T_x of tangent directions at x is in one-to-one correspondence with the connected components of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{x\}$, and that:

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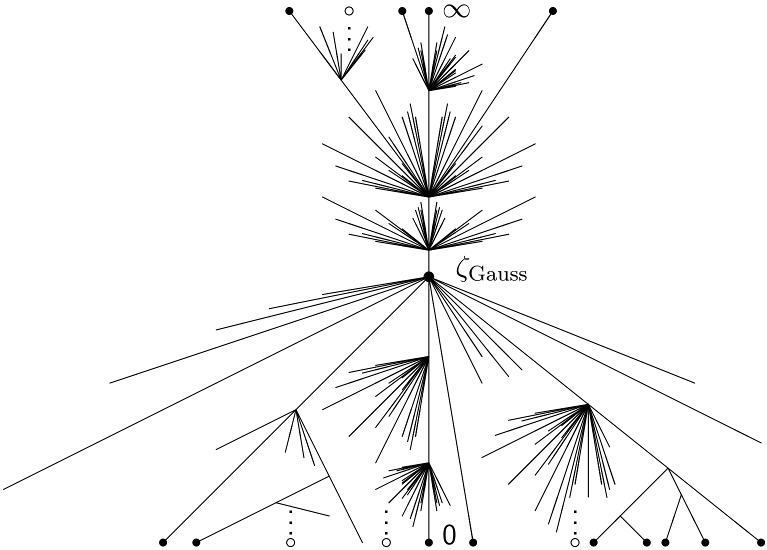
$$|T_x| = \begin{cases} |\mathbb{P}^1(\tilde{K})| & x \text{ of type II} \\ 2 & x \text{ of type III} \\ 1 & x \text{ of type I or type IV.} \end{cases}$$

The Berkovich projective line



Continuous piecewise affine functions

- Let U be a connected open subset of $\mathbb{P}^1_{\mathrm{Berk}}$ (with respect to the Berkovich topology), and let $f : U \to \mathbb{R} \cup \{\pm\infty\}$ be an extended-real valued function which is finite-valued on $\mathbf{H}_{\mathrm{Berk}}$.

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   $t \mapsto a_{\vec{v}} t + b_{\vec{v}}$.)
3. For each $x \in U \cap \mathbf{H}_{\mathrm{Berk}}$, we have $a_{\vec{v}} = 0$ (i.e., f is constant along sufficiently short paths emanating from x in the direction $\vec{v}$) for all but finitely many $\vec{v} \in T_x$.

Directional derivatives

- If $x \in U \cap \mathbf{H}_{\mathrm{Berk}}$ and $f \in \mathrm{CPA}(U)$, then for each $\vec{v} \in T_x$, the directional derivative $d_{\vec{v}} f(x)$ is well-defined.

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- In particular, the quantity

$$\Delta_x(f) := -\sum_{v \in T_x} d_{\vec{v}} f(x)$$

is well-defined.

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   - $x \in \mathbb{P}^1(K)$ and h is constant on an open neighborhood of x.

Example: $\log^+ |x|$

Example

- Consider the function $G : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$G(x) = \begin{cases} +\infty & x = \infty \\ \log_v \max(|T|_x, 1) & x \in \mathbb{A}^1_{\mathrm{Berk}}. \end{cases}$$

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- The restriction of G to K is $\log_v^+ |x| = \log_v \max(|x|, 1)$.
- G is harmonic on $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta_{\mathrm{Gauss}}, \infty\}$.

Example: $\log^+|x|$ (continued)

Indeed, let Λ denote the closed path from ζ_{Gauss} to ∞ in $\mathbb{P}^1_{\mathrm{Berk}}$, and let $r_\Lambda : \mathbb{P}^1_{\mathrm{Berk}} \twoheadrightarrow \Lambda$ be the natural retraction map from $\mathbb{P}^1_{\mathrm{Berk}}$ onto Λ. Then:

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1. G is not harmonic at $\zeta_{\rm Gauss}$: the sum of the slopes of G in all directions emanating from $\zeta_{\rm Gauss}$ is 1.

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Remark

1. G is not harmonic at ζ_{Gauss}: the sum of the slopes of G in all directions emanating from ζ_{Gauss} is 1.
2. G is also not harmonic at ∞.

Example: $\log|f|$ for f analytic and nowhere zero

Example

Let $V = \mathcal{M}(\mathcal{A}_V)$ be an affinoid subdomain of $\mathbb{P}^1_{\mathrm{Berk}}$, and let U be a connected open subset of V. If $f \in \mathcal{A}_V$ is a nowhere zero analytic function on V, then the function $\log_v |f|_x$ is harmonic on U.

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This generalizes the well-known classical fact that if f is a nowhere zero analytic function on an open subset U of the complex plane, then $\log |f|$ is harmonic on U.

The maximum principle

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2. *If h is a harmonic function on a domain $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ which extends continuously to the closure $\bar{U}$ of U, then h achieves both its minimum and maximum values on the boundary ∂U of U.*

A consequence of the maximum principle

Recall that a simple domain in $\mathbb{P}^1_{\mathrm{Berk}}$ is a connected open set $V \subseteq \mathbb{P}^1_{\mathrm{Berk}}$ whose boundary is a finite subset of $\mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$.

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Every harmonic function on a simple domain V in $\mathbb{P}^1_{\mathrm{Berk}}$ extends continuously to $\bar{V}$.

If $V = \mathbb{P}^1_{\mathrm{Berk}}$, then then ∂V is empty, and if V is a Berkovich open disk, then ∂V consists of a single point. By the second part of the maximum principle, we conclude:

A consequence of the maximum principle

Recall that a simple domain in $\mathbb{P}^1_{\mathrm{Berk}}$ is a connected open set $V \subseteq \mathbb{P}^1_{\mathrm{Berk}}$ whose boundary is a finite subset of $\mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$.

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Corollary

If $U = \mathbb{P}^1_{\mathrm{Berk}}$ or U is an open Berkovich disk, then every harmonic function on U is constant.

The main dendrite of a domain

An important observation is that the behavior of a harmonic function on a domain $U \subseteq \mathbb{P}^1_{\mathrm{Berk}}$ is controlled by its behavior on a certain special subset.

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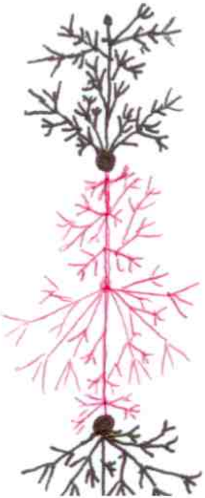
For example:

- If U is a Berkovich open disk, then $D(U)$ is empty.
- If $U = \mathcal{B}(a, R)^{-} \backslash \mathcal{B}(a, r)$ is a Berkovich open annulus, then $D(U)$ is the open segment joining the two boundary points $\zeta_{a,r}$ and $\zeta_{a,R}$ of U.

A Berkovich open annulus



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Example

If $K = \mathbb{C}_p$ and $U = \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(\mathbb{Q}_p)$, then the main dendrite $D(U)$ is a locally finite real tree in which the set of branch points is discrete, and every branch point has degree $p + 1$. In fact, $D(U)$ can be identified with the (geometric realization of the) Bruhat-Tits tree for $\mathrm{PGL}(2, \mathbb{Q}_p)$.

Harmonic functions and the main dendrite

Theorem

Let U be a domain in $\mathbb{P}^1_{\mathrm{Berk}}$, and let h be harmonic on U. If the main dendrite is empty, then h is constant; otherwise, h is constant on all branches leading away from the main dendrite.

The Poisson Formula (classical case)

- In the classical theory of harmonic functions in the complex plane, if f is harmonic on an open disk V then it has a continuous extension to the closure of V, and the Poisson Formula expresses the values of f on V in terms of its values on the boundary of V.

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- Specifically, if $V \subseteq \mathbb{C}$ is an open disk of radius r centered at z_0, and if f is harmonic in V, then f extends continuously to $\bar{V}$ and $f(z_0) = \int_{\partial V} f \, d\mu_V$, where μ_V is the uniform probability measure $d\theta/2\pi$ on the boundary circle ∂V.

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$$f(z) = \int_{\partial V} f \, d\mu_{z,V}$$

for every harmonic function f on $\bar{V}$.

The Poisson Formula for $\mathbb{P}^1_{\mathrm{Berk}}$

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- In $\mathbb{P}^1_{\text{Berk}}$, the basic open neigbhorhoods are the simple domains, which have only a finite number of boundary points.
- Every harmonic function f on a simple domain V has a continuous extension to its closure.
- There is an analogue of the Jensen-Poisson measure which yields an explicit formula for f on V in terms of its values on the boundary of V. In other words, one can explicitly solve the Berkovich space analogue of the Dirichlet problem on any simple domain.

The Gromov product

Definition

- For $x, y, z \in \mathbf{H}_{\mathrm{Berk}}$, define the Gromov product $(x|y)_z$ by

$$(x|y)_z = \rho(w, z),$$

where w is the first point where the unique paths from x to z and y to z intersect.

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Remark

This definition plays an important role in Gromov's theory of δ-hyperbolic spaces, with $\mathbf{H}_{\mathrm{Berk}}$ being an example of a 0-hyperbolic space.

Some linear algebra

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Lemma

For each $z \in V \cap \mathbf{H}_{\mathrm{Berk}}$, there is a unique probability vector $[h_1(z), \ldots, h_m(z)]$ for which the quantity

$$h_1(z)(x_i|x_1)_z + \cdots + h_m(z)(x_i|x_m)_z$$

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is independent of i.

One can give an explicit formula for $h_1(z), \ldots, h_m(z)$ using Cramer's rule.

Harmonic measures

- For each i, the function $z \mapsto h_i(z)$, defined originally for $z \in V \cap \mathbf{H}_{\mathrm{Berk}}$, extends by continuity to a map $h_i : \bar{V} \to \mathbb{R}$, called the i^{th} harmonic measure with respect to V.

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- h_i is a harmonic function on V.
- The values of h_i on ∂V are given by the formula $h_i(x_j) = \delta_{ij}$.
- By construction, we have $0 \leq h_i(z) \leq 1$ for all $z \in \bar{V}$ and $h_1 + \cdots + h_m \equiv 1$ on $\bar{V}$.

The Poisson formula

Theorem (Poisson formula)

Let V be a simple domain in $\mathbb{P}^1_{\mathrm{Berk}}$ with boundary points $x_1, \ldots, x_m$. Then each harmonic function f on V has a continuous extension to $\bar{V}$, and there is a unique such function with a prescribed set of boundary values

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Let V be a simple domain in $\mathbb{P}^1_{\mathrm{Berk}}$ with boundary points $x_1, \ldots, x_m$. Then each harmonic function f on V has a continuous extension to $\bar{V}$, and there is a unique such function with a prescribed set of boundary values Moreover, f can be computed from its boundary values using the formula

$$f(z) = \sum_{i=1}^{m} f(x_i) \cdot h_i(z),$$

valid for all $z \in \bar{V}$.

The Jensen-Poisson measure

- For $z \in V$, define the Jensen-Poisson measure $\mu_{z,V}$ on V relative to the point z by

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Corollary

If V is a simple domain in $\mathbb{P}^1_{\mathrm{Berk}}$, then a continuous function $f : \bar{V} \to \mathbb{R}$ is harmonic in V if and only if

$$f(z) = \int_{\partial V} f \, d\mu_{z,V}$$

for all $z \in V$.

Limits of harmonic functions

The Poisson formula can be used to prove that any limit of a sequence of harmonic functions is harmonic, under a much weaker condition than is required classically.

Theorem

Let U be a domain in $\mathbb{P}^1_{\mathrm{Berk}}$. Suppose $f_1, f_2, \ldots$ are harmonic in U and converge pointwise to a function $f : U \to \mathbb{R}$. Then $f(z)$ is harmonic in U, and the $f_i(z)$ converge uniformly to $f(z)$ on compact subsets of U.

Harnack's principle

There is also a Berkovich space analogue of Harnack's principle:

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1. $\lim_{i\to\infty} f_i \equiv +\infty$; *or*
2. $f(z) = \lim_{i\to\infty} f_i(z)$ *is finite for all z, the $f_i(z)$ converge uniformly to $f(z)$ on compact subsets of U, and $f(z)$ is harmonic in U.*

Subharmonic functions

Definition

Let $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ be a domain. A function $f : U \to [-\infty, \infty)$ with $f(x) \not\equiv -\infty$ is called subharmonic on U if

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(SH2) For each simple subdomain V of U (i.e., a simple domain V whose closure is contained in U), we have

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for all $z \in V$.

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f is called superharmonic on U if $-f$ is subharmonic on U.

Subharmonic functions (continued)

Remark

1. By the Poisson formula, condition (SH2) can be replaced by the condition that for each simple subdomain $V \subset U$ and each harmonic function h on V, if $f(x) \leq h(x)$ on ∂V then $f(x) \leq h(x)$ on V.

Subharmonic functions (continued)

Remark

1. By the Poisson formula, condition (SH2) can be replaced by the condition that for each simple subdomain $V \subset U$ and each harmonic function h on V, if $f(x) \leq h(x)$ on ∂V then $f(x) \leq h(x)$ on V.
2. f is harmonic on U if and only if it is both subharmonic and superharmonic on U.

Examples of subharmonic functions

Example

Let $V = \mathcal{M}(\mathcal{A}_V)$ be an affinoid subdomain of $\mathbb{P}^1_{\mathrm{Berk}}$, and let U be a connected open subset of V. If $f \in \mathcal{A}_V$ is analytic on V, then the function $\log_v |f|_x$ is subharmonic on U.

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Example

For fixed $y, z \in \mathbf{H}_{\mathrm{Berk}}$, the function $f(x) = (x|y)_z$ is superharmonic in $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{z\}$, and subharmonic in $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{y\}$.

Maximum principle for subharmonic functions

Theorem (Maximum Principle)

1. *If f is a nonconstant subharmonic function on a domain $U \subset \mathbb{P}^1_{\mathrm{Berk}}$, then f does not achieve a global maximum on U.*

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2. *If f is a subharmonic function on a domain $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ which extends continuously to $\bar{U}$, then f achieves its maximum value on ∂U.*

Subharmonic functions and the main dendrite

The following result shows that at any given point, there are only finitely many tangent directions in which a subharmonic function can be increasing:

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The following result shows that at any given point, there are only finitely many tangent directions in which a subharmonic function can be increasing:

Theorem

Let f be subharmonic on a domain U. Then f is non-increasing on paths leading away from the main dendrite of U. If U is a disk, then f is non-increasing on paths leading away from the unique boundary point of U.