Potential Theory on Berkovich Spaces
Lecture 5: Potential theory on Berkovich Curves

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2. Potential theory on Berkovich curves has been extensively developed in the recent Ph.D. thesis of Amaury Thuillier; we will only scratch the surface of his theory.
3. This lecture assumes significantly more background in rigid analysis and Berkovich's global theory of K-analytic spaces than the previous lectures did.

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- X_{Berk}: The Berkovich K-analytic space corresponding to X.

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- We denote by π the reduction map $\pi : X(K) \to Z(\tilde{K})$.

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2. *If z is an ordinary double point of Z, then $X(z)$ is analytically isomorphic to the standard open annulus of height α for a unique $\alpha \in |K^*|$ with $0 < \alpha < 1$.*

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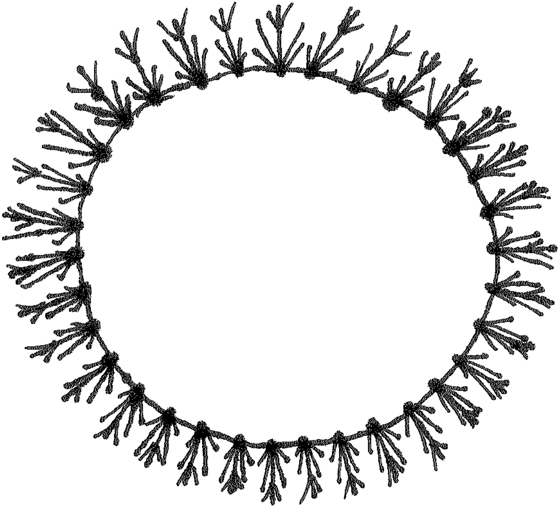
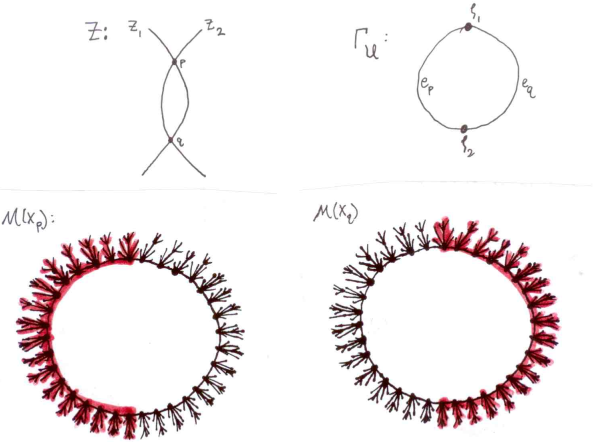
- Key facts:
  1. Γ_Z can be naturally endowed with the structure of a metrized graph.
  2. There is a natural inclusion map $\iota_{\mathfrak{X}} : \Gamma_Z \hookrightarrow X_{\mathrm{Berk}}$, and a natural deformation retraction $r_{\mathfrak{X}} : X_{\mathrm{Berk}} \twoheadrightarrow \Gamma_Z$.

Example: A Tate elliptic curve



Pasting together a Tate elliptic curve



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- For each irreducible component Z_i of Z, there is a unique point ζ_i of X_{Berk} reducing to the generic point of Z_i.

$\mathbf{H}(X)$ and its canonical metric

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- In particular, if p is a double point of Z and $X(p) \cong A(\alpha)^-$, then the length of the edge e_p connecting ζ_i and ζ_j is $-\log_v |\alpha|$, the modulus of the open annulus $A(\alpha)^-$.

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- One can show that the metric on $\mathbf{H}(X)$ is canonical (i.e., it does not depend on our choice of a particular semistable model $\mathfrak{X}$ for X).

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- If $g(X) \geq 2$ and X is defined over a discretely valued subfield K_0 of K with valuation ring R_0, the skeleton of X_{Berk} is the dual graph associated to the minimal regular model of X over R_0.

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- We can view the Berkovich analytic space E_{Berk} as the space obtained from the closed annulus $V = \mathcal{B}(0,1) \backslash \mathcal{B}(0,|q|)^-$ by identifying the affinoid subspaces $V_1 = \mathcal{B}(0,1) \backslash \mathcal{B}(0,1)^-$ and $V_2 = \mathcal{B}(0,|q|) \backslash \mathcal{B}(0,|q|)^-$.

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- If $\pi : V \to E_{\mathrm{Berk}}$ denotes the corresponding quotient map, then π is an isometry and the closed segment of length $-\log|q|$ between the two boundary points of V gets mapped onto the skeleton of E_{Berk}, which is a circle.
- $\mathbb{P}^1_{\mathrm{Berk}}\backslash\{0,\infty\}$ is the universal covering space of E_{Berk} via the natural map $\mathbb{P}^1_{\mathrm{Berk}}\backslash\{0,\infty\} \twoheadrightarrow E_{\mathrm{Berk}}$, with covering group $\mathbb{Z}$.

- For $x \in X_{\mathrm{Berk}}$, define the set T_x of tangent directions at x to be the set of equivalence classes of paths emanating from x.

Continuous piecewise-affine functions on Berkovich curves

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1. The restriction of f to $\mathbf{H}(X)$ is piecewise-affine with respect to the path metric ρ.
2. For each $x \in U \cap \mathbf{H}(X)$, we have $d_{\vec{v}} f(x) = 0$ for all but finitely many $\vec{v} \in T_x$.

For $f \in \mathrm{CPA}(U)$, the quantity

$$\Delta_x(f) := -\sum_{\vec{v} \in T_x} d_{\vec{v}} f(x)$$

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    - $x \in \mathbf{H}(X)$ and $\Delta_x(h) = 0$; or
    - $x \in X(K)$ and h is constant on an open neighborhood of x.

Functions of the form $-\log|f|$

Theorem

Let f be a nonzero meromorphic function on X. Let $F(x)$ be the unique continuous map from X_{Berk} to $\mathbb{R} \cup \{\pm\infty\}$ extending the function $-\log_v |f(x)|$ on $X(K)$. Then $F \in \mathrm{CPA}(X_{\mathrm{Berk}})$, and F is harmonic on $X_{\mathrm{Berk}} \setminus \mathrm{Div}(f)$.

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- Every point $x \in X_{\mathrm{Berk}} \backslash \{\zeta_1, \ldots, \zeta_t\}$ has an open neighborhood isomorphic to a simple domain in $\mathbb{P}^1_{\mathrm{Berk}}$.
- By the corresponding result for $\mathbb{P}^1_{\mathrm{Berk}}$, in order to show that F is harmonic outside the zeros and poles of f, it suffices to check that $\Delta_{\zeta_i}(F) = 0$ for each $i = 1, \ldots, t$.

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- We will now explain why F is harmonic at points $\zeta_1, \ldots, \zeta_t$.

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- We make this bijection explicit by writing

$$T_{\zeta_i} = \{\vec{v}(z)\}_{z \in Z_i(\tilde{K})},$$

where $\vec{v}(z)$ is a formal unit vector emanating from ζ_i in the direction corresponding to $z \in Z_i(\tilde{K})$.

Orders of vanishing

- For each component Z_i, there exists a nonzero scalar $c_i = c_i(f) \in K^*$, depending on Z_i and on f, such that $c_i^{-1}f$ reduces to a nonzero rational function on Z_i.

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- The absolute value $|c_i|$ of c_i, and the divisor of $\overline{c_i^{-1} f}$ on Z_i, depend only on f.

Orders of vanishing

- For each component Z_i, there exists a nonzero scalar $c_i = c_i(f) \in K^*$, depending on Z_i and on f, such that $c_i^{-1}f$ reduces to a nonzero rational function on Z_i.
- The absolute value $|c_i|$ of c_i, and the divisor of $\overline{c_i^{-1}f}$ on Z_i, depend only on f.
- We can thus define the order of f at a point $z \in Z_i(\tilde{K})$ to be

$$\mathrm{ord}_{z,Z_i}(f) := \mathrm{ord}_z(\overline{c_i^{-1}f}).$$

Tangent spaces and orders of vanishing

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Let f be a nonzero meromorphic function on X, and let $z \in Z(\tilde{K})$. Then for each $i = 1, \ldots, t$, the directional derivative $d_{\vec{v}(z)}(\log_v |f|)(\zeta_i)$ equals $\mathrm{ord}_{z,Z_i}(f)$.

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This "explains" why $-\log_v |f|$ is harmonic at ζ_i.

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Theorem

$(X_{\mathrm{Berk}}, \{\Gamma_X\})$ is an arboretum, and in particular, X_{Berk} is homeomorphic to $\varprojlim_{\mathfrak{X}} \Gamma_{\mathfrak{X}}$.

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Theorem (Poincaré-Lelong formula)

Let f be a nonzero meromorphic function on X, and let $F \in \mathrm{CPA}(X_{\mathrm{Berk}})$ be the unique continuous map from X_{Berk} to $\mathbb{R} \cup \{\pm\infty\}$ extending the function $-\log_v |f(x)|$ on $X(K)$. Then

$$\Delta_{X_{\mathrm{Berk}}}(F) = \delta_{\mathrm{Div}(f)} \ .$$

The canonical distance relative to a point

- If X_{Berk} is a Berkovich curve and $y, z \in X_{\mathrm{Berk}}$, there is, up to a multiplicative constant, a unique function $[x, y]_z$ on $\mathbb{P}^1_{\mathrm{Berk}}$ for which $f(x) = -\log_v [x, y]_z : X_{\mathrm{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$ belongs to $\mathrm{BDV}(X_{\mathrm{Berk}})$ and satisfies

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- This generalizes the canonical distance $[x, y]_\infty$ on $\mathbb{A}^1_{\mathrm{Berk}}$, which was introduced in the first lecture, and which extends the function $|x - y|$ on K.

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Factorizing absolute values of meromorphic functions

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Theorem

If f is a nonzero meromorphic function on X with divisor $\mathrm{Div}(f) = \sum m_i(a_i)$, then for any $z \in X_{\mathrm{Berk}}$ there is a constant C (depending on z and f) such that

$$|f|_x = C \cdot \prod [x, a_i]_z^{m_i}$$

for all $x \in X_{\mathrm{Berk}}$.

Local heights on elliptic curves

- Let E_{Berk} be the Berkovich analytic space associated to an elliptic curve E/K, and let Σ be the skeleton of E_{Berk}, which is a point $\{\zeta\}$ if E has good reduction, and is a circle of length $\ell = -\log_v |q| > 0$ if E has multiplicative reduction.

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- In the case where E is defined over a discretely valued subfield K_0 of K with valuation ring R_0, ζ is the unique point of E_{Berk} reducing to the generic point of the special fiber of the Néron model of E over R_0.

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- The standard Néron canonical local height function $\lambda : E(K) \backslash \{O\} \to \mathbb{R}$ extends naturally to a function $\lambda : E_{\mathrm{Berk}} \to \mathbb{R} \cup \{+\infty\}$ which is singular only at the origin O.

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- This shows that λ can be considered as a kind of *Green's function* on E_{Berk}, just as in the archimedean case.
- With the appropriate definitions, λ is subharmonic on $E_{\mathrm{Berk}} \backslash \{O\}$ and superharmonic on $E_{\mathrm{Berk}} \backslash \Sigma$.

A global application

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Equidistribution of small points

The following is an "adelic" version of the Szpiro-Ullmo-Zhang equidistribution theorem for elliptic curves.

Theorem (Chambert-Loir, Baker–Petsche)

Let k be a number field, and let E/k be an elliptic curve. Fix a place $v \in M_k$ and an embedding $E(\bar{k}) \hookrightarrow E(\mathbb{C}_v) \subseteq E_{\mathrm{Berk},v}$. Suppose that $\{P_n\}$ is a sequence of distinct points in $E(\bar{k})$ such that $\hat{h}(P_n) \to 0$, and let δ_n be the probability measure on $E_{\mathrm{Berk},v}$ supported equally on the set of $\mathrm{Gal}(\bar{k}/k)$-conjugates of P_n. Then $\delta_n \to \mu_v$ weakly on $E_{\mathrm{Berk},v}$.

The energy minimization principle

This theorem can be proved using Arakelov theory (as developed in the non-archimedean case by Chambert-Loir and Thuillier), or in a more “elementary” way by means of the following energy minimization principle:

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Theorem

Let v be a place of k, let $\mu = \mu_v$, and define the "energy functional" I on the space $\mathbb{P}$ of probability measures on $E_{\mathrm{Berk},v}$ by the formula

$$I(\nu) \ := \ \iint_{E_{\mathrm{Berk},v} \times E_{\mathrm{Berk},v}} \lambda_v(x,y)\, d\nu(x) d\nu(y).$$

Then $I(\nu) \geq I(\mu)$ for all $\nu \in \mathbb{P}$, with equality if and only if $\nu = \mu$.