Non-Archimedean Geometry

Lectures from a course by Matt Baker at U.C. Berkeley

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Non-Archimedean Fields

Definition 1. An valued field is a field $k$ together with a non-zero absolute value $|\cdot| : k \to \mathbb{R}_{\geq 0}$, satisfying

1. $|0| = 0, |1| = 1$
2. $|xy| = |x||y|$
3. $|x + y| \leq |x| + |y|$

A non-archimedean valued field is a valued field that additionally satisfies the stronger condition

$3' \; |x + y| \leq \max(|x|, |y|)$

An archimedean valued field is a valued field that is not non-archimedean.

We will almost always assume that $k$ is complete, i.e. if we define a metric $d$ on $k$ by $d(x, y) := |x - y|$ then $k$ is a complete metric space with respect to $d$.

Example 1. $\mathbb{R}$ and $\mathbb{C}$ are complete archimedean valued fields. In fact, it is a non-trivial theorem that they are the only complete archimedean valued fields. (This is a consequence of a theorem by Gelfand and Mazur; see Theorem 1.2.3 of Engler and Prestel, “Valued Fields”.)

Example 2. $\mathbb{Q}_p$, the field of $p$-adic numbers, is a complete non-archimedean valued field.

Aside: if you don’t know what $\mathbb{Q}_p$ is, chances are this course is not at your level and you should consider not taking it. However, the definition is reviewed here nonetheless.
**Definition of** $\mathbb{Q}_p$: Let $p$ be a prime and $0 < \epsilon < 1$ (definition is independent of choice of $\epsilon$; by convention we usually take $\epsilon = 1/p$).

For $n \in \mathbb{Z}$, define $\text{ord}_p(n) := k$ iff $p^k || n$ (|| denotes “exactly divides”).

For $\alpha = \frac{a}{b} \in \mathbb{Q}$ define $\text{ord}_p(\alpha) := \text{ord}_p(a) - \text{ord}_p(b)$.

Let $| \cdot |_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ be the valuation defined by $|\alpha|_p := \epsilon^{\text{ord}_p(\alpha)}$ for $\alpha \in \mathbb{Q}$.

Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to $| \cdot |_p$.

Then $\mathbb{Q}_p$ is a complete non-archimedean valued field and $\mathbb{Q}$ is dense in $\mathbb{Q}_p$.

**Example 3.** Let $F$ be any field and denote by $F((t))$ the field of Laurent series in $t$ over $F$, i.e. elements of $F((t))$ are of the form $\sum_{n=N}^{\infty} a_n t^n$ with $N \in \mathbb{Z}$, $a_N \neq 0$.

Then we have an absolute value $| \cdot | : F((t)) \to \mathbb{R}_{\geq 0}$ defined by $| \sum_{n=N}^{\infty} a_n t^n | := \epsilon^N$, and $F((t))$ is complete with respect to $| \cdot |$.

**Example 4.** It is a non-trivial theorem that $\mathbb{C}_p$, the completion of $\overline{\mathbb{Q}}_p$ (algebraic closure of $\mathbb{Q}_p$), is complete and algebraically closed.

Aside: later we will see that $\mathbb{C}_p$ is “not big enough” and we will bring in the notions of Berkovich affine line and spherical closure to remedy this.

**Example 5.** For a field $F$, the field of Puiseux Series over $F$ is defined by

$$\bigcup_{n \in \mathbb{N}} F((t^{1/n}))$$

so that the elements of the field of Puiseux Series over $F$ are series of the form

$$\sum_{k=k_0}^{\infty} a_k t^{k/n}$$

where $k_0, n \in \mathbb{Z}, n > 0,$ and $a_k \in F$

If $F$ is algebraically closed and characteristic zero, then the field of Puiseux Series over $F$ is also algebraically closed. However, the field of Puiseux Series over $F$ is not complete.
Geometry over $k$

Let $X$ be an algebraic variety over $k$, an algebraically closed, complete, non-archimedean valued field with non-trivial absolute value (the trivial absolute value is that defined by $|0| = 0$ and $|x| = 1$ for all $x \neq 0$).

We assign to $X(k)$ the weakest topology such that for every Zariski open subset $U$ of $X$ and every regular function $f$ in $O_X(U)$, the function $U(k) \to \mathbb{R}$ defined by $P \mapsto |f(P)|$ is continuous.

**Problem:** with this topology, $X(k)$ is totally disconnected and not locally compact.

**Solution:** Two options:

1. Generalize the notion of topology and covering (Grothendick topology)
2. Add more points to “locally compactify” (Berkovich’s idea)

Sketch of solution 2: we will construct an object $X^{an}$ with the following properties:

- $X^{an}$ is locally compact and Hausdorff
- $X$ is path connected iff $X$ is connected
- $X(k) \hookrightarrow X^{an}$ and $X(k)$ is dense in $X^{an}$
- $X^{an}$ is compact iff $X$ is proper
- The topological dimension of $X^{an}$ is equal to the Krull dimension of $X$

We will define $X^{an}$ in future lectures; for now we provide a picture of $\mathbb{A}^1^{an}$. 


Figure 1: The Berkovich affine line (illustration courtesy of Joe Silverman)
Analogy to Understand Berkovich Affine Line

Consider

\[ X := [0, 1] \cap \mathbb{Q} = \{ x \in \mathbb{Q} \mid 0 \leq x \leq 1 \} \]

with the subspace topology inherited from the metric topology on the interval \([0, 1]\). It is totally disconnected and not locally compact.

We want to compactify \(X\) by first defining the “correct” notion of continuous functions on \(X\). For example, following Gelfand we can define the completion of \(X\) to be the spectrum of the algebra of continuous functions on \(X\), given the “right” definition of continuous.

**Naive Definition.** Let \(f : X \to \mathbb{Q}\) be continuous at \(x \in X\) if for all \(\epsilon > 0\) there exists a \(\delta > 0\) such that for all \(x' \in X\), we have \(|f(x') - f(x)| < \epsilon\) whenever \(|x - x'| < \delta\).

**Problem With Naive Definition:** We get continuous functions that can’t be extended to continuous functions on the reals.

For example, let \(z \in [0, 1] \setminus \mathbb{Q}\) and define a function \(f : X \to \mathbb{R}\) by

\[
    f(x) := \begin{cases} 
    0 & x < z \\
    1 & x > z
    \end{cases}
\]

Then there is no way to assign a value \(f(z)\) to \(z\) such that \(f\) becomes a continuous function on \([0, 1]\).

**Solutions:**

1. Tate-Grothendick: restrict the notion of open set and open cover
   
   In this analogy, define open sets to be intervals of the form \((a, b)\) with \(a, b \in \mathbb{Q}\) and open covers to be finite unions of open sets. With this definition, \(X\) becomes compact and connected.
2. Berkovich: complete $X$

In this analogy, replace $X$ with the interval $[0, 1]$ and the usual notion of open set and open cover.

**Berkovich Affine Line**

Given a complete valued field $k$, we would like construct a notion of affine line $\mathbb{A}^1_k$ such that

- $\mathbb{A}^1_k = \mathbb{C}$
- The ring of algebraic functions on $\mathbb{A}^1_k$ is $k[T]$.

To this end, we consider all multiplicative semi-norms on $k[T]$ that extend the value $|\cdot|$ given on $k$.

Recall: a *multiplicative semi-norm* on a ring $A$ is a function $|\cdot| : A \to \mathbb{R}_{\geq 0}$, satisfying

1. $|0| = 0, |1| = 1$
2. $|xy| = |x||y|$
3. $|x + y| \leq |x| + |y|$

If $k$ is a valued field and $A$ is a $k$-algebra, a *$k$-semi-norm* on $A$ is a semi-norm on $A$ whose restriction to $k$ is the given absolute value on $k$.

Suppose $k$ is a complete, algebraically closed, non-archimedean valued field ($\mathbb{C}_p$ is a good example to keep in mind). For every $z \in k$, we have the evaluation semi-norm defined by:

$$|\cdot|_z : k[T] \to \mathbb{R}_{\geq 0}$$

$$f \mapsto |f(z)|$$

Then $|\cdot|_z \neq |\cdot|_y$ when $z \neq y$, so identifying $z \in k$ with $|\cdot|_z$ gives an injection $k \hookrightarrow \{\text{multiplicative } k\text{-semi-norms on } k[T]\}$.

**Definition 1.** Let $\mathbb{A}^1_{\text{Berk}}$ be the set of multiplicative $k$-semi-norms on $k[T]$ endowed with the weakest topology such that for all $f \in k[T]$, the map

$$\mathbb{A}^1_{\text{Berk}} \to \mathbb{R}_{\geq 0}$$

$$||\cdot|| \mapsto ||f||$$

is continuous.
Note that when $k = \mathbb{C}$, the Gelfand-Mazur theorem tells us that all multiplicative semi-norms on $k[T]$ are of the form $|f| = |f(z)|$ for some $z \in \mathbb{C}$.

**Exercise 1.** Prove that if $k = \mathbb{C}$ then the injection $k \hookrightarrow \mathbb{A}_1^{\text{Berk}}$ is a homeomorphism $k$ onto its image.

If $k = \mathbb{C}_p$ then $\mathbb{A}_1^{\text{Berk}}$ has many more points than those arising from $k$. For example, given $a \in k$ and $r \in \mathbb{R}_{\geq 0}$, let $D(a,r) := \{z \in k \mid |z - a| \leq r\}$ be a closed disk in $k$. Define

$$|| \cdot ||_D : k[T] \to \mathbb{R}_{\geq 0}$$

$$f \mapsto \sup_{z \in D} |f(z)|$$

**Exercise 2.** Show that $|| \cdot ||_D$ is a multiplicative semi-norm on $k[T]$

**Hint:** If $f = \sum_{i=0}^n a_i(T - a)^i$ then $||f||_D = \max_{0 \leq i \leq n} |a_i|r^i$ since $| \cdot |$ is non-archimedean.

Note: The norm $|| \cdot ||_{D(0,1)}$ is often called the Gauss norm.

**Exercise 3.** Show that the map $D(a,r) \mapsto || \cdot ||_{D(a,r)}$ yields an imbedding from the set of generalized closed disks of $k$ into $\mathbb{A}_1^{\text{Berk}}$ that extends the injection $k \hookrightarrow \mathbb{A}_1^{\text{Berk}}$ by associating an element $a$ of $k$ with the generalized disk $D(a,0)$.

Now we can begin to understand the tree-like picture of the Berkovich affine line (below). Each point on the tree is a point of $\mathbb{A}_1^{\text{Berk}}$, i.e. a multiplicative semi-norm on $k[T]$. The vertical axis measures the radius of the closed disk associated to the point. On the horizontal axis are the points of $\mathbb{A}_1^{\text{Berk}}$ corresponding to disks of radius zero; as we saw these are in bijection with elements of $k$. If $a,b \in k$ and $|b - a| \leq r$ then $\{z \in k \mid |z - a| \leq r\} = \{z \in k \mid |z - b| \leq r\}$. Thus the line segments starting at $a$ and $b$ indicate points corresponding to disks of increasing radius centered at $a$ and $b$ respectively, and at some radius $r = |a - b|$, the disks become *the same disk*; here the two segments meet.
Theorem 0.1. The Berkovich affine line is uniquely path-connected.

Proof. See Lemma 2.10 of “Potential Theory and Dynamics on the Berkovich Projective Line” (Baker and Rumley).

Note that what we have drawn is not locally compact as we have not yet identified all the points of \( \mathbb{A}^1_{\text{Berk}} \). In fact, there are multiplicative semi-norms on \( k[T] \) that are not of the form \( || \cdot ||_D \) for a closed disk \( D \) of \( k \).

For example, let \( k = \mathbb{C}_p \) and let \( \mathcal{D} = D_1 \supset D_2 \supset D_3 \ldots \) be a sequence of nested closed disks of \( k \). If \( D_1 \cap D_2 \cap \ldots = \emptyset \) then \( || \cdot ||_{\mathcal{D}} \), defined by \( ||f||_{\mathcal{D}} := \lim_{n \to \infty} ||f||_{D_n} \), is a new point of \( \mathbb{A}^1_{\text{Berk}} \), i.e. a multiplicative semi-norm on \( k[T] \) that is not equal to \( || \cdot ||_D \) for any closed disk \( D \). We call points arising in this fashion “type 4 points”. Types 1 – 3 are points coming from generalized closed disks \( D(a, r) \) and are characterized as follows:

Type 1: \( r = 0 \) (equivalently, these are points of \( k \))

Type 2: \( r \in |k^\times| \)

Type 3: \( r \notin |k| \)

Theorem 0.2. (Berkovich) When \( k \) is algebraically closed, complete, and non-archimedean, all points of \( \mathbb{A}^1_{\text{Berk}} \) are of types 1 – 4.
(Throughout this day’s notes we’ll assume that $k$ is a complete, algebraically closed, non-Archimedian field. Also, a point in $\mathbb{A}^1_{\text{Berk},k}$ will sometimes be written as $x$ and sometimes as $|\cdot|_x$.)

Points of $\mathbb{A}^1_{\text{Berk},k}$

Recall 1. For $k$ a complete, algebraically closed, non-Archimedian field, the Berkovich affine line $(\mathbb{A}^1_k)^{an}$ (sometimes denoted $\mathbb{A}^1_{\text{Berk},k}$ or $\mathbb{A}^1_{\text{Berk}}$), is defined (as a set) as

\[ \{ \text{multiplicative seminorms on } k[T] \text{ extending } |\cdot| \text{ on } k \} \]

If $D = D(a, r) := \{ x \in k \mid |x - a| \leq r \}$, then

\[ |f|_D := \sup_{z \in D} |f(z)| \]

is a multiplicative seminorm.

**Theorem 0.1** (Berkovich). Every point $x \in (\mathbb{A}^1_k)^{an}$ can be realized as

\[ \lim_{n \to \infty} |\cdot|_{D_n} \]

for some nested sequence $D_1 \supseteq D_2 \supseteq \ldots$ of closed disks in $k$.

**Lemma 0.2.** If $k$ is a non-Archimedean field, $A$ is a $k$-algebra, and $|\cdot|$ is a multiplicative seminorm on $A$ extending the absolute value on $k$, then for all $f, g \in A$,

\[ |f + g| \leq \max\{|f|, |g|\}, \]

with $|f + g| = |g|$ if $|f| < |g|$. 
Proof of Lemma. This proof is essentially a clever application of the binomial theorem. In particular, we know that $|m| = |1 + \ldots + 1| \leq |1| = 1$ for all integers $m$.

$$|f + g|^n = |(f + g)^n| = \sum_{k=0}^{n} \binom{n}{k} f^k g^{n-k} \leq \sum_{k=0}^{n} \binom{n}{k} f^k g^{n-k} \leq (n + 1) \max\{ |f|, |g| \}.$$  

Taking $n$th roots and letting $n \to \infty$, we have the desired result.

For the case of $|f| < |g|$, we know that $|f + g| \leq |g|$. On the other hand, $|g| = |g + f - f| \leq \max\{|f + g|, |f|\}$. Since it is not the case $|g| \leq |f|$, we know that $|g| \leq |f + g|$. Thus $|g| = |f + g|$.

Proof of Theorem. We will prove the theorem in the case of plugging in any linear polynomial $T - a$ of $k[T]$ into the seminorm, rather than an arbitrary polynomial. This result, combined with $k = \overline{k}$ and the multiplicativity of our seminorms, will give the desired result.

Fix $x \in \mathbb{A}^1_{\text{Berk}}$. Consider the family $\mathcal{F}$ of closed disks

$$\mathcal{F} := \{ D(a, |T - a|_x) \mid a \in k \}.$$  

Claim 1: $\mathcal{F}$ is totally ordered by inclusion.

To establish this claim, let $a, b \in k$. Without loss of generality, $|T - a|_x \geq |T - b|_x$. We have

$$|a - b| = |a - b|_x = |(T - b) - (T - a)|_x \leq \max\{|T - a|_x, |T - b|_x\} \quad (*)$$  

by the lemma, with equality if $|T - a|_x > |T - b|_x$. Thus $b \in D(a, |T - a|_x)$, implying that $D(b, |T - b|_x) \subset D(a, |T - a|_x)$, establishing Claim 1.

Now, let $r = \inf_{a \in k} |T - a|_x$. Choose a sequence of points $a_n$ such that $r_n$ decreases to $r$, with $r_n = |T - a_n|_x$. Let $D_n = D(a_n, r_n)$.

Claim 2: For all $a \in k$, $|T - a|_x = \lim_{n \to \infty} |T - a|_{D_n}$.

To prove this claim, choose any $a \in k$. By definition of $r$, $|T - a|_x \geq r$.

We will deal with two cases.

1. Assume $|T - a|_x = r$. Then $r_n = |T - a_n|_x \geq |a_n - a|$ by $(*)$ (taking $a = a_n$ and $b = a$). Hence $a \in D_n$, so $|T - a|_{D_n} = \sup_{z \in D_n} |z - a| = r_n$.

   It follows that

   $$|T - a|_x = r = \lim_{n \to \infty} r_n = \lim_{n \to \infty} |T - a|_{D_n},$$

   as desired.
(2) Assume $|T - a|_x > r$. For $n \gg 0$, $|T - a|_x > |a_n|_x = r_n$. Applying the strict case of $(\ast)$, this implies that $|T - a|_x = |a - a_n|$ for $n \gg 0$, and therefore that $|a - a_n| > |T - a_n|_x = r_n$ for $n \gg 0$. Hence $|T - a|_{D_n} = \sup_{z \in D_n} |z - a| = |a_n - a| = |T - a|_x$ for sufficiently large $n$. This lets us conclude that $\lim |T - a|_{D_n} = |T - a|_x$, as desired.

This establishes Claim 2, as well as the theorem.

\[ \square \]

Remark 1. In the preceding proof, we can choose the radii of the disks $D_n$ to lie in the value group of $k$.

The four types of points in $\mathbb{A}^1_{\text{Berk}}$ To describe the four types of points in the Berkovich affine line, for a point $|\cdot|_x$ we’ll let $\{D_n\}$ be the corresponding descending sequence of closed disks, writing $D_n = D(a_n, r_n)$ and $r = \lim r_n$.

- Type I: These points, which correspond to points of $k$, have $|f|_x = |f(a)|$ for some $a \in k$. (That is, $r = 0$.)
- Type II: These points have
  \[ \bigcap_{n=1}^{\infty} D_n = D(a, r) \]
  with $r \in |k^\times|$ (the value group of $k$).
- Type III: These points have
  \[ \bigcap_{n=1}^{\infty} D_n = D(a, r) \]
  with $r \notin |k|$.
- Type IV: These points have
  \[ \bigcap_{n=1}^{\infty} D_n = \emptyset \]
  with $r > 0$ by the completeness of $k$. 

3
Local description of the points  For $x \in \mathbb{A}^1_{\text{Berk}}$, let

\[ T_x = \{ \text{tangent directions at } x \} = \{ \text{connected components of } \mathbb{A}^1_{\text{Berk}} - \{x\} \}. \]

- If $x$ is a Type I point, then $|T_x| = 1$.
- If $x$ is a Type II point, then there are infinitely many branches at that point. In particular, the branches are in noncanonical bijection with $\mathbb{P}^1(\bar{k})$ (to be made more precise later), so $|T_x| = |\mathbb{P}^1(\bar{k})|$.  
- If $x$ is a Type III point, then there is no branching, and $|T_x| = 2$.
- Type IV points are sort of “dead ends”, giving $|T_x| = 1$.

The topology on this infinitely branched tree, the observer’s topology, is a very weak topology which makes the tree locally compact. For a basis of fundamental open neighborhoods, take the connected components of $\mathbb{A}^1_{\text{Berk}} - \{ \text{a finite set of points} \}$. 
Facts about non-Archimedean fields

1. If $k$ is a complete non-Archimedean field and $K/k$ is an algebraic extension, then there exists a unique extension to $K$ of $|\cdot|$ on $k$. If $K/k$ is a finite extension, the $K$ is complete.

2. If $k$ is a non-Archimedean field which is algebraically closed, then its completion $\hat{k}$ is also algebraically closed (and complete, of course). Moreover, $k$ is dense in $\hat{k}$ and $\hat{k}/k$ is an immediate extension (meaning that $|\hat{k}| = |k|$ and $\hat{k}$ and $k$ have the same residue field).

Theorem 0.1. $\mathbb{Q}_p$ is not complete.

Sketch of Proof. Define $X_n = \{ x \in \mathbb{Q}_p | [\mathbb{Q}_p(x) : \mathbb{Q}_p] = n \}$. Let $X = \mathbb{Q}_p$, so that $\bigcup_{n=1}^{\infty} X_n = X$.

One may check that each $X_n$ is closed and has empty interior. Recall the Baire Category Theorem: in a complete metric space, any countable union of closed sets with empty interior has empty interior (and so can’t be the whole space). This implies that $X$ is not complete.

Definition 1. We say $k$ is spherically complete if every descending nested sequence of closed disks has nonempty intersection.

Remark 1. Spherically complete implies complete. However, the converse is not true, by the following theorem.

Theorem 0.2. $\mathbb{C}_p := \hat{\mathbb{Q}}_p$ is complete and algebraically closed but not spherically complete.

Proof. Let $r > 0$, and let $r_n$ be a descending sequence with $r_n \to r$, so that $r_0 > r_1 > \ldots > \lim r_n = r$, where $r_n \in |\mathbb{Q}_p|$. Consider $D_0 = D(0, r_0)$. Choose disjoint disks $D_1$ and $D_1'$ of radius $r_1$ contained in $D$. Continue this fashion, giving us a tree $T$ that keeps branching off in pairs. For each
infinite (rooted) path in $T$, we get a nested sequence $D_\gamma$ of closed disks. Let $D_\gamma = \bigcap_{D \in D_\gamma} D$.

We claim that each $D_\gamma$ is either empty or is a closed disk of radius $r > 0$. Assume $D_\gamma$ is nonempty, and let $x \in D_\gamma$. Since $x$ is in each $D \in D_\gamma$, it is the center of each disk (by the ultrametric property), and so each $D$ contains $D(x, r)$, meaning $D(x, r) \subset D_\gamma$. However, if $|x - y| = r' > r$, there is some $D \in D$ of radius $r_i < r'$, meaning $y \notin D$, so $y \notin D_\gamma$. Thus $D_\gamma$ is a disk of radius $r$, or it is empty.

In particular, each $D_\gamma$ is open (since $r > 0$). By construction the $D_\gamma$’s are disjoint. Since $\mathbb{C}_p$ contains $\mathbb{Q}$ as a countable dense subset, only countably many $D_\gamma$’s can be nonempty by disjointness. There were uncountably many paths, so uncountably many $D_\gamma$’s are empty. We conclude that $\mathbb{C}_p$ is not spherically complete.

Construction of a spherically complete extension of $k$ Our construction is based on ultrafilters.

**Definition 2.** Let $X$ be a set. A filter on $X$ is a nonempty collection $\mathcal{F}$ of subsets of $X$ such that

1. $\emptyset \notin \mathcal{F}$.
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
3. If $A \in \mathcal{F}$ and $A \subset B \subset X$, then $B \in \mathcal{F}$.

**Example 1.** Choose $x \in X$. Define $\mathcal{F}_x = \{ A \subset X \mid x \in A \}$.

**Example 2.** If $X$ is infinite, let $\mathcal{F} = \{ A \subset X \mid |A^C| < \infty \}$. This is called the Fréchet filter.

Filters are ordered by refinement.

**Definition 3.** We say that $\mathcal{F}'$ refines $\mathcal{F}$, denoted $\mathcal{F}' \supset \mathcal{F}$, if $\mathcal{F}' \supset \mathcal{F}$ in $P(X)$.

**Definition 4.** A maximal filter with respect to this relation is called an ultrafilter.

Recommended reading for ultrafilters: Terry Tao’s blog.

**Lemma 0.3.** A filter $\mathcal{F}$ is an ultrafilter if and only if for all $A \subset X$, exactly one of $A$ and $A^C$ belongs to $\mathcal{F}$. 2
By Zorn’s lemma, every filter is contained in an ultrafilter.

**Definition 5.** A filter of the form $\mathcal{F}_x$ is called *principal*. Such a filter is always an ultrafilter.

**Remark 2.** Arrow’s Impossibility Theorem says that every filter of a finite set is principal (i.e., of the form $\mathcal{F}_x$).

On any infinite set, there exist non-principal ultrafilters. For instance, take a maximal filter containing the Fréchet filter.

Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then any bounded sequence $\{a_n\}$ of real numbers has a unique “$\mathcal{U}$-limit.” This can be constructed by doing a binary search, dividing some bounded interval containing $\{a_n\}$ and zooming in on the half with $a_i$’s with indices forming an element of $\mathcal{U}$.

Given a non-Archimedean field $k$, we offer a construction of a field $\Omega_k$ that is algebraically closed and spherically complete.

Fix $\mathcal{U}$. Let $R$ be the normed ring $\ell^\infty(k)$ (i.e., bounded infinite sequences in $k$ with the $\ell^\infty$ norm). Define $\varphi : R \to \mathbb{R}_{\geq 0}$ by $\varphi((a_n)) = \lim_{\mathcal{U}} |a_n|$. Let $J = \varphi^{-1}(0)$, which is an ideal in $R$. Let $\Omega_k = R/J$.

**Theorem 0.4.** 1. $(\Omega_k, |\cdot|)$ is a non-Archimedean field containing $k$.

2. $\Omega_k$ is spherically complete.

3. If $k$ is algebraically closed, then so is $\Omega_k$.

4. If $|k|$ is dense in $\mathbb{R}$, then $|\Omega_k| = \mathbb{R}$.

Reference: Chapter 3 of Robert’s “A Course in $p$-adic Analysis.”
Recall: spherically complete extensions of $k$. Given $k$, a non-Archimedean field, we constructed a non-Archimedean field $\Omega$ extending $k$ such that $\Omega$ is spherically complete (as well as algebraically closed if $k$ is algebraically closed).

The $\Omega$ we constructed is in general not an immediate extension of $k$. If $k = \mathbb{F}$ with nontrivial valuation, when the value group $|k|$ is dense in $\mathbb{R}$, meaning that $|\Omega_k| = \mathbb{R}$. For instance, in the case of $k = \mathbb{C}_p$, we have $|k^\times| = p^\mathbb{Q}$, but $|\Omega_k^\times| = \mathbb{R}^\times$. Immediate extensions preserve value groups, so this extension is not immediate.

There is a minimal spherically closed field extension of $k$ inside of $\Omega$. All such extensions are isomorphic, but not canonically; in particular, the isomorphisms might not preserve $k$. (Reference: van Rooij, “Non-archimedean Functional Analysis.”)

Analytification of Affine Varieties over $k$. We’ll take $k$ to be complete and non-Archimedean, but maybe not algebraically closed.

An affine variety $X/k$ corresponds to a reduced finitely generated $k$-algebra $A$. From this we will construct a topological space $X^{an}$, which we’ll also write as $M(A)$.

**Definition 1.** As a set, we define

$$X^{an} := \{\text{multiplicative seminorms on } A \text{ extending } | \cdot | \text{ on } k\}.$$

(For concision we will call such seminorms $k$-multiplicative seminorms.)

The topology is the weakest such that all maps $\psi_f : X^{an} \to \mathbb{R}$ are continuous, where $\psi_f$ is defined for each $f \in A$ by $x \mapsto |f|_x$.

Equivalently, this is the subspace topology induced by $X \hookrightarrow \mathbb{R}^A$, $x \mapsto (f \mapsto |f|_x)$ (where $\mathbb{R}^A$ has the product topology).

Some facts about $X^{an}$:

- The analytification $X \to X^{an}$ is functorial.
• One can glue analytifications and thereby analytify any algebraic variety.

• Later on, we will put more structure on $X^\text{an}$ and will be able to analytify more general objects, such as schemes of finite type over $k$, or more generally quasiseparated and quasicompact rigid analytic spaces over $k$.

A motivating recollection from algebraic geometry  Usually we define

$$\text{Spec } A := \{\text{prime ideals in } A\}.$$  

We can instead define

$$\text{Spec } A := \{\text{$k$-algebra homomorphisms } A \to K, \text{ where } K/k \text{ is an extension}\}/\sim,$$

where the equivalence relation $\sim$ is given by $\varphi \sim \varphi'$ if and only if we have an injection $K \hookrightarrow K'$ such that $A \to K \hookrightarrow K'$ agrees with $A \to K'$.

This is equivalent to the prime ideal definition by the maps

$$\mathfrak{p} \to [A \to \text{Frac}(A/\mathfrak{p})]$$

$$\ker \varphi \leftarrow (A \xrightarrow{\varphi} K)$$

Say we’re in the case $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Then we can think of the points of $\text{Spec } A$ as “solutions” to $f_1, \ldots, f_m$ with values in some field $K/k$.

(One interpretation of this is in terms of transcendental versus algebraic elements. For instance, in $\text{Spec } k[x, y]/(y)$, our points look like $(x, 0)$. Maybe $x$ is algebraic, or maybe it’s transcendental. All transcendental extensions $k(T)$ are equivalent, so the generic point classifies the transcendental points.)

An alternate definition of $X^\text{an}$. We’ll consider

$$\chi(A) := \{\text{$k$-algebra homomorphisms } A \to K\}/\sim$$

where $K$ is a complete valued field extending $| \cdot |$ on $k$, and where equivalence is the same as for the algebraic geometry case, except with the requirement that $K \hookrightarrow K'$ is compatible with $| \cdot |$.

Claim 0.1. $\chi(A)$ is naturally in bijection with $M(A) = X^\text{an}$.
This bijection is given by the maps

$$| \cdot |_x : (A \to \mathcal{H}(x))$$

$$|f| := |\varphi(f)| \leftarrow (A \xrightarrow{\varphi} K),$$

where we define $\mathcal{H}(x)$ to be the completion of $\text{Frac}(A/\ker(| \cdot |_x))$; we call $\mathcal{H}(x)$ the completed residue field of $x$.

**Classification of Points on $\mathbb{A}^1_{\text{Berk}}$ in Terms of $\mathcal{H}(x)$’s**  
(See Proposition 2.3 in Baker-Rumely.)

Assume that $k$ is algebraically closed for simplicity. Let $A = k[T]$, and $x \in \mathbb{A}^1_{\text{Berk}}$. Let $k(x) = \text{Frac}(A/\ker(| \cdot |_x))$, the algebraic residue field of $x$. By definition, $\mathcal{H}(x) = \widetilde{k(x)}$.

- For Type I points, $k(x) = \mathcal{H}(x) = k$.

- For points of Type II, III, and IV, $k(x) \cong k(T)$, and $\mathcal{H}(x)$ is a weird completion. It can be described explicitly for Type III, but not for Types II and IV.

| Type | Value Group $|\mathcal{H}(x)\times|$ | Residue field $\widetilde{\mathcal{H}(x)}$ |
|------|-------------------------------|----------------------------------|
| I    | $|k^\times|$                  | $\widetilde{k}$                  |
| II   | $|k^\times|$                  | $\widetilde{k(T)}$               |
| III  | $\langle |k^\times|, r \rangle$, $r \notin |k^\times|$ | $k$                             |
| IV   | $|k^\times|$                  | $\widetilde{k}$                  |

(Note that we have $|\mathcal{H}(x)\times| = |k(x)\times|$ and $\widetilde{\mathcal{H}(x)} = \widetilde{k(x)}$.)
Classification of Berkovich points in terms of $H(x)$

Recall that $K(x)$ is defined to be the fraction field of $k[T]/\ker(\cdot|_x)$, and $H(x)$ is defined to be its completion $\widehat{K(x)}$, with respect to the absolute value on $K(x)$ induced by the multiplicative seminorm $|\cdot|_x$ on $k[T]$.

**Theorem.** Let $k$ be a complete and algebraically closed non-Archimedean field. Then for each point $x \in \mathbb{A}^1_{\text{Berk}, k}$, its completed residue field $\widehat{H(x)}$ has one of the following descriptions:

| Type | Value group $|H(x)|$ | Residue field $\widehat{H(x)}$ |
|------|---------------------|-------------------------------|
| I    | $|k^*|$              | $k$                           |
| II   | $|k^*|$              | $\bar{k}(t)$                 |
| III  | $(|k^*|, r)$         | $\bar{k}$                    |
| IV   | $|k^*|$              | $\bar{k}$                    |

where $r \in \mathbb{R} \setminus |k^*|$ in the value group of Type III above. Moreover, $x$ is of Type I if and only if $H(x) \cong k$.

**Remark.** A point $x$ is of Type IV if and only if $H(x)$ is a nontrivial immediate extension of $k$, that is, it is not equal to $k$ but it has the same value group and residue field as $k$.

**Proof.** We give a sketch of the proof. For more details, see Proposition 2.3 in [1].

We consider the four types of points separately.

**Type I:** In this case, $x$ corresponds to a point $a \in k$, and $|f|_x = |f(a)|$. Then $\ker(\cdot|_x = m_a = (T-a)$, and so $H(x) = K(x) = k$.

**Type II:** In this case, $x$ corresponds to a disk $D(a, r)$ where $r = |c|$ for some $c \in k^*$. Then $K(x) \cong k(T)$ since the kernel of $|\cdot|_x$ must be 0.

Now, $H(x)$ is the completion of $k(T)$ with respect to the norm $|\cdot|_x$ on $k[T]$. By the non-Archimedean maximum principle, for each $f \in k[T]$ there exists a $p \in D(a, r)$ such that $|f|_x = |f(p)|$. Hence, we have $|k(T)|_x = |k^*|_x$.

Let $t$ be the reduction of $(T-a)/c$ in $H(x)$. (Exercise: Prove that $t$ is transcendental over $\bar{k}$.)

Then we have $\widehat{H(x)} = \widehat{K(x)} = \bar{k}(t)$.

**Type III:** In this case, the residue field is $\bar{k}$ because all polynomials in $T$ with $|\cdot| \leq 1$ have constant reduction to $\bar{k}$. We have $|T-a|_x = r \notin |k^*|$. This implies that $|H(x)|^* = (|k^*|, r)$.

**Type IV:** In this case, $x$ corresponds to a nested collection of disks $D(a_i, r_i)$ with empty intersection. If $f \in k(T)$ is nonzero, there exists an $N$ such that $D_N := D(a_N, r_N)$ contains no zeros or poles of $f$. This implies that $|f|$ is constant on $D_N$, and so $|f|_x = |f|_{D(a_N, r_N)} \in |k^*|$. It follows that $|H(x)|^* = |k^*|$

Now, let $f = g/h$, $|f|_x = 1$. For $N >> 0$, we have $|g|_x = |g|_{D_N} = |h|_x = |h|_{D_N}$. It follows that $f \equiv g(a_N)/h(a_N) \mod m_x$.

It follows that $\bar{f} \in \bar{k}$.

\[\square\]
References

Inequalities in non-Archimedean field extensions

**Algebraic extensions:**

Let \( K/k \) be a finite extension of non-Archimedean valued fields. Let \( e = [v(K^*) : v(k^*)] \) and \( f = [\tilde{K} : \tilde{k}] \). Then \( ef \leq [K : k] \) \((1)\). We sketch a proof below. Equality holds if \( k \) is discretely valued.

**Proof.** Choose \( x_1, \ldots, x_e \in K^* \) such that \( v(x_1), \ldots, v(x_e) \) represent distinct cosets in \( v(K^*)/v(k^*) \). Choose \( y_1, \ldots, y_f \in K^0 \) such that \( \tilde{y}_1, \ldots, \tilde{y}_f \) in \( \tilde{K} \) are linearly independent over \( \tilde{k} \). It suffices to prove the following:

**Claim:** The elements \( x_i y_j \), where \( 1 \leq i \leq e \) and \( 1 \leq j \leq f \), are linearly independent over \( k \).

To prove the claim, assume \( \sum a_{ij} x_i y_j = 0 \) \((2)\) for some \( a_{ij} \in k \) not all zero. Choose \((i_0, j_0)\) such that \( v(a_{i_0 j_0} x_{i_0} y_{j_0}) \) is minimal. We show that any other monomial term with \( v(a_{ij} x_i y_j) \) minimal must also have \( i = i_0 \). To see this, note that \( v(y_j) = 0 \), and if \( v(a_{i_0 j_0} x_{i_0} y_{j_0}) = v(a_{i_1 j_1} x_{i_1} y_{j_1}) \) then using the relation \( v(xy) = v(x) + v(y) \), we obtain

\[
\begin{align*}
0 &= v(x_{i_0}) - v(x_{i_1}) - v(a_{i_1 j_1}) + v(a_{i_0 j_0}) \\
      &= \left( v(a_{i_1 j_1}) - v(a_{i_0 j_0}) \right) + \left[ \left( v(x_{i_0}) - v(x_{i_1}) \right) - v(a_{i_0 j_0}) \right].
\end{align*}
\]

Since the \( x_i \) were chosen so that \( v(x_1), \ldots, v(x_e) \) represent \( e \) distinct cosets in \( v(K^*)/v(k^*) \), it follows that \( i_1 = i_0 \).

Finally, multiplying \((2)\) by \((a_{i_0 j_0} x_{i_0})^{-1} \), we get a relation of the form \( \sum b_j y_j + c = 0 \) where the \( b_j \)'s are in \( k \) with \( |b_j| \leq 1 \), \( c \in \tilde{K} \) with \( |c| < 1 \), and \( |b_{j_0}| = 1 \). Passing to the residue field, we get a nontrivial relation in the \( \tilde{y}_j \)'s, which contradicts the assumption that the \( \tilde{y}_j \)'s are linearly independent over \( \tilde{k} \).

We can now prove a similar inequality for the transcendental case.

**Transcendental extensions:**

Suppose \( K/k \) is a finitely generated extension of non-Archimedean valued fields. Let \( s = \dim_{Q} \left( \frac{v(K^*)}{v(k^*)} \otimes_{Z} Q \right) \) and \( t = \text{tr. deg}(\bar{K}/\bar{k}) \). Then we have

\[
s + t \leq \text{tr. deg}(K/k) \tag{3}\]

This is known as Abhyankar’s Inequality.
Proof. (Sketch) Choose \( x_1, \ldots, x_s \in K^* \) such that the \([v(x_i)] \otimes 1\) are linearly independent over \( \mathbb{Q} \). Choose \( y_1, \ldots, y_t \in K^0 \) such that \( \tilde{y}_1, \ldots, \tilde{y}_t \) are algebraically independent over \( k \). It suffices to show:

Claim: The elements \( x_1, \ldots, x_s, y_1, \ldots, y_t \) are algebraically independent over \( k \).

To prove the claim, assume for the sake of contradiction that \( p \) is not identically zero. Then \( p(x, y) = 0 \) for some \( p \in k[X_1, \ldots, X_s, Y_1, \ldots, Y_t] \).

We can then expand this relation into a sum of monomial terms, and consider their valuations. Choosing a term of minimal valuation, by the same argument as above, we find that all of the subscripts of \( x \) must be equal, and we can again divide out by that. We then get a polynomial relation in the \( \tilde{y}_i \)’s, a contradiction. \( \square \)

Classification of points of \( \mathbb{A}_\text{Berk}^1 \)

We can make the following table of values of \( s \) and \( t \), for \( K(x) \), in the one-dimensional case.

<table>
<thead>
<tr>
<th>Type</th>
<th>( s )</th>
<th>( t )</th>
<th>( \text{tr. deg}(K(x)/k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Remark 1. There is a similar classification of points for nonsingular analytic curves.

We now take a look at the two-dimensional case, \( \mathbb{A}_\text{Berk}^2 \):

- Every nested sequence of (possibly degenerate) closed polydisks gives a \( k \)-multiplicative seminorm on \( R = k[X, Y] \).
- Assume \( k \) is trivially valued. Every algebraic curve \( C \) in \( \mathbb{A}_k^2 \) gives a discrete valuation on \( R \) given by \( \text{ord}_C(f) \). (This is the “divisorial valuation.”) For such valuations we have \( s = 1 \) and \( t = 1 \).
- When \( k \) is trivially valued, we also get a different sort of point of \( \mathbb{A}_\text{Berk}^2 \) from germs of arbitrary analytic curves: for example, let \( \tau \in \tilde{R} \) be transcendental over \( R \), and write \( \tau = x + \sum_{i=1}^{\infty} c_i y^i \) where \( c_i \in k^* \) for all \( i \). Then the map
  \[
  \phi : R \to k[[y]]
  \]
  sending \( y \mapsto y \) and
  \[
  x \mapsto x - \tau = - \sum c_i y^i
  \]
  is an injective homomorphism. In this case, we have \( s = 1 \) and \( t = 0 \).

We can then restrict the \( y \)-adic valuation on \( k[[y]] \) to \( R \).
Today we are going to talk about the spectrum of a Banach ring.

Let $\mathcal{A}$ be a commutative ring with identity (as all rings in this class, unless otherwise specified). We then define a norm on $\mathcal{A}$ to be a function $\| \cdot \| : \mathcal{A} \to \mathbb{R}_{\geq 0}$ such that:

1. $\|0\| = 0$ and $\|1\| = 1$.
2. $\|f\| = 0$ iff $f = 0$.
3. $\|fg\| \leq \|f\| \cdot \|g\|$.
4. $\|f + g\| \leq \|f\| + \|g\|$.

It is interesting to note that the spectrum of the zero ring is empty but that the zero ring is still considered a normed ring.

A Banach ring is a pair $(\mathcal{A}, \| \cdot \|)$ such that $\mathcal{A}$ is a commutative right with identity, $\| \cdot \|$ is a norm on $\mathcal{A}$, and $\mathcal{A}$ is complete with respect to this norm.

Examples of Banach rings:

1. Every field $k$ which is complete with respect to some absolute value $| \cdot |$ is a Banach ring.
2. $(\mathbb{Z}, | \cdot |_{\infty})$ is a Banach ring.
3. (Most important example for this course) Let $k$ be a complete non-archimedean valued field. Define the Tate algebra to be

$$T_n := k\langle T_1, ..., T_n \rangle := \{ \sum_{\nu \in \mathbb{N}^n} a_\nu T^\nu : \lim_{|\nu| \to \infty} |a_\nu| = 0 \} \subseteq k[[T_1, ..., T_n]]$$

and the Gauss norm to be $\| \sum_{\nu \in \mathbb{N}^n} a_\nu T^\nu \| = \max_{\nu \in \mathbb{N}^n} |a_\nu|$. The Tate algebra together with the Gauss norm is a Banach ring.

Note that Berkovich denotes the Tate algebra by $k\{T_1, ..., T_n\}$ and uses the notation $k\langle T_1, ..., T_n \rangle$ for something different... which we will get to next lecture, but will not need for the rest of the course. We are using the notation $k\langle T_1, ..., T_n \rangle$ to denote the Tate algebra as this is the notation used in Bosch’s notes.
The (Berkovich) spectrum $\mathcal{M}(A)$ of a Banach ring $A$ is the set of all nonzero bounded multiplicative seminorms on $A$, where by a bonded multiplicative seminorm we mean a nonzero function $|\cdot| : A \to \mathbb{R}_{\geq 0}$ such that:

1. $|fg| = |f| \cdot |g|.$
2. $|f + g| \leq |f| + |g|.$
3. There exists $C > 0$ such that $|f| \leq C \|f\|$ for all $f \in A$

Note that condition 3 for a seminorm is equivalent to $f$ being continuous on $A$. The topology on $\mathcal{M}(A)$ is the weakest one such that $|\cdot|_x \mapsto |f|_x \in \mathbb{R}$ is continuous for all $f \in A$.

It should be clear from the context if $A$ is a Banach ring or a $k$-algebra, where $k$ is a valued field, as our definition of $\mathcal{M}(A)$ is slightly different in these two contexts. We now have the following theorem about $\mathcal{M}(A)$, whose proof we will cover in this lecture and the next:

**Theorem 0.1.** (Berkovich) If $A$ is a nonzero Banach ring, then $\mathcal{M}(A)$ is a non-empty compact Hausdorff space.

In order to prove this theorem, we will need a couple of lemmas:

**Lemma 0.2.** If $x \in \mathcal{M}(A)$ then $|f|_x \leq \|f\|$ for all $f \in A$.

**Proof.** We have that $(|f|_x)^n = |f^n|_x \leq C \|f^n\| \leq C \|f\|^n$. So we have $|f|_x \leq \sqrt[n]{C}\|f\|$ for all $n$. Taking the limit as $n \to \infty$ gives the desire result. 

In order to understand the statements of the next two lemmas, we need to know the definition of a net (which is different than Berkovich’s notion of a net). First, we need the notion of a directed set; a directed set $I$ is a set together with a reflexive, transitive, binary relation $\leq$ such that for all $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $\alpha \leq \gamma, \beta \leq \gamma$.

**Examples** of directed sets:

1. $(\mathbb{N}, \leq)$.
2. Let $X$ be a topological space. For $x \in X$, the set $I = \{\text{Open nbds of } x\}$ with the relation $U \leq V \iff V \subseteq U$ is a directed set.

A net in a set $X$ is a map $I \to X$, where $I$ is a directed set. We write $\langle X_\alpha \rangle$ instead of $\alpha \mapsto X_\alpha$. Note that a sequence is the same thing as a net indexed by the natural numbers. We say that $X_\alpha$ converges to $X$ if for every neighborhood $U$ of $X$, there exists $\alpha_0 \in I$ such that $X_\alpha \in U$ whenever $\alpha \geq \alpha_0$.

**Lemma 0.3.** A subset $Y$ of a topological space $X$ is closed iff whenever $\langle Y_\alpha \rangle$ is a net in $Y$ converging to $x \in X$, we have $x \in Y$.

**Lemma 0.4.** A map $f : X \to Y$, where $X, Y$ are topological spaces, is continuous iff $f(X_\alpha) \to f(X)$ whenever $X_\alpha \to X$. 
Let us now begin to prove our theorem by showing that $\mathcal{M}(\mathcal{A})$ is compact. Consider the following map:

$$i : \mathcal{M}(\mathcal{A}) \to T := \prod_{f \in \mathcal{A}} [0, \|f\|], \ x \mapsto (|f|_x)_f$$

By definition(s), $i$ is injective and $\mathcal{M}(\mathcal{A})$ is homeomorphic to its image. By Tychonoff’s theorem, $T$ is compact and Hausdorff. Thus, it suffice to prove that $\mathcal{M} := i(\mathcal{M}(\mathcal{A}))$ is closed.

So suppose we have a net $\langle X_\alpha \rangle$ in $\mathcal{M}(\mathcal{A})$ converging to some $x \in T$. We want to show that $x \in \mathcal{M}$. This is equivalent to showing that if $|\cdot|_\alpha$ is a net in $\mathcal{M}(\mathcal{A})$ such that $|f|_\alpha \to |f|_x$ for all $f \in \mathcal{A}$, then $|\cdot|_x$ is a bounded multiplicative seminorm. This is clear.

Next lecture, we will show that $\mathcal{M}(\mathcal{A}) \neq \emptyset$, discuss $\mathcal{M}(\mathbb{Z})$, and $\mathbb{A}^1_{Berk,k}$, where $k$ is trivially valued.

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Let \((A, \| \cdot \|)\) be a Banach ring. We defined the Berkovich spectrum \(\mathcal{M}(A)\) to be the set of all bounded multiplicative seminorms on \(A\), i.e., functions \(| \cdot | : A \to \mathbb{R}_{\geq 0}\) such that

\begin{itemize}
  \item \(|0| = 0\) and \(|1| = 1\),
  \item \(|fg| = |f||g|\) and \(|f+g| \leq |f| + |g|\) for every \(f, g \in A\), and
  \item \(|f| \leq \|f\|\) for every \(f \in A\).
\end{itemize}

Our goal is to show that \(\mathcal{M}(A)\) is nonempty if \(A\) is nonzero. This will require some preparation.

**Definition 1.** If \((A, \| \cdot \|)\) is a Banach ring and \(I\) is an ideal, the **residue seminorm** on \(A/I\) is the norm given by

\[
|\pi(a)| = \inf_{a' \in A, \pi(a')=\pi(a)} \|a'\| = \inf_{x \in I} \|a-x\|,
\]

for \(a \in A\), where \(\pi : A \to A/I\) is the canonical map.

**Lemma 1.** If \(I\) is a proper ideal \((1 \notin I)\), then the residue seminorm \(| \cdot |\) on \(A/I\) is a seminorm. If \(I\) is closed, then the residue seminorm is a norm on \(A/I\), making it into a Banach ring.

**Proof.** For any \(a \in A\), we have \(|\pi(a)| \leq \|a\|\). Therefore, \(|0| = 0\) and \(|1| \leq \|1\| = 1\). If \(|1| \neq 1\), then \(||1-x|| < 1\) for some \(x \in I\). But then \(||(1-x)^k|| \leq ||1-x||^k\) by submultiplicativity of \(| \cdot |\), and therefore \(\sum_{k=0}^{\infty} ||(1-x)^k|| < \infty\). Then the series

\[
1 + (1-x) + (1-x)^2 + \cdots
\]

converges absolutely to some value \(y \in A\). Multiplying by \((1-x)\), we see that \((1-x)y = (1-x)+(1-x)^2+\cdots = y-1\). Rearranging \((1-x)y = y-1\) gives \(xy = 1\). Then \(x \in I\) implies that \(xy = 1 \in I\), a contradiction. So \(|1| = 1\).

To check submultiplicativity and the triangle law, let \(f\) and \(g\) be two elements of \(A/I\). Let \(f_1, f_2, \ldots\) and \(g_1, g_2, \ldots\) be sequences of elements of \(A\) with \(\pi(f_i) = f\) and \(\pi(g_i) = g\) for every \(i\), and \(\lim_{i \to \infty} ||f_i|| = ||f||\) and \(\lim_{i \to \infty} ||g_i|| = ||g||\). Then \(\pi(f_i g_i) = f g\) and \(\pi(f_i + g_i) = f + g\), so

\[
|fg| \leq \lim_{i \to \infty} ||f_i g_i|| \leq \lim_{i \to \infty} ||f_i|| \cdot ||g_i|| = |f| \cdot |g|
\]

and

\[
|f + g| \leq \lim_{i \to \infty} ||f_i + g_i|| \leq \lim_{i \to \infty} ||f_i|| + ||g_i|| = |f| + |g|.
\]

Finally, suppose that \(I\) is closed. If \(\pi(a) \in A/I\) has \(|\pi(a)| = 0\), then by definition there are \(x \in I\) with \(||a-x||\) arbitrarily small. Since \(I\) is closed, \(a \in I\) and \(\pi(a) = 0\). Thus \(| \cdot |\) is a norm. Completeness of
A/M can be verified as follows. Let \( \pi_1, \pi_2, \ldots \) be a Cauchy sequence. We only need to show that some subsequence converges. Choose \( n_1 < n_2 < \cdots \) such that \( |\pi_{n_i} - \pi_{n_{i+1}}| \leq 2^{-i} \) for every \( i \). Then we can inductively choose \( b_i \in A \) such that \( \pi(b_i) = \pi_{n_i} \) and \( |b_i - b_{i+1}| \leq 2^{1-i} \). Then \( \lim_{i \to \infty} b_i \) converges since \( \sum_{i>0} |b_i - b_{i+1}| < \infty \). Let \( b = \lim_{i \to \infty} b_i \). Then \( |\pi_{n_i} - \pi(b)| \leq |b_i - b| \), so \( \lim_{i \to \infty} \pi_{n_i} = \pi(b) \). This in turn makes \( \lim_{i \to \infty} \pi_i = \pi(b) \in A/M \). Thus \( A/M \) is complete. \( \square \)

**Lemma 2.** Let \( (A, || \cdot ||) \) be a Banach ring. Then every maximal ideal \( M \subseteq A \) is closed.

**Proof.** Any nonzero element \( a \in A/M \) has a multiplicative inverse, because \( A/M \) is a field. But then \( 1 = |1| = |aa^{-1}| \leq |a| \cdot |a^{-1}| \), which implies that \( |a| \neq 0 \). This shows that the residue seminorm on \( A/M \) is a norm. Therefore the topology on \( A/M \) from the residue seminorm is Hausdorff, and \( \{0\} \) is a closed subset of \( A/M \). Now since \( |\pi(a)| \leq ||a|| \) for any \( a \in A \), the map \( A \rightarrow A/M \) is uniformly continuous.\(^1\) Since \( \pi \) is continuous and \( \{0\} \) is closed, \( \pi^{-1}(\{0\}) = M \) is also a closed set.

Alternatively, we could note that the closure of any ideal is again an ideal. As noted in the proof of Lemma 2, every \( x \in A \) with \( ||1-x|| < 1 \) is invertible. Therefore, \( 1 \) is in the interior of the set of invertible elements, so the closure of any proper ideal is again a proper ideal. Therefore, if \( M \) is maximal, then \( M \subseteq \overline{M} \) implies that \( M = \overline{M} \). \( \square \)

**Lemma 3.** If \( A \) is a normed ring, and \( f \in A \) is invertible, then for \( m, n \geq 0 \)

\[
||f^n||^{-m} \leq ||f^{-m}||^n.
\]

**Proof.** By submultiplicativity,

\[
||(f^n)^m|| \leq ||f^n||^m
\]

and

\[
||f^{nm}|| \cdot ||f^{-nm}|| \geq ||f^{nm} f^{-nm}|| = ||1|| = 1.
\]

Therefore,

\[
||f^n||^{-m} \leq ||f^{nm}||^{-1} \leq ||f^{-nm}|| \leq ||f^{-m}||^n.
\]

The first inequality is the reciprocal of (2), the second inequality follows from (2), and the third inequality follows directly from submultiplicativity. \( \square \)

**Lemma 4.** If \( \sum_i a_i < \infty \) and the \( a_i \) are positive numbers, then \( \sum_i a_i^n < \infty \) for any \( n \geq 1 \).

**Proof.** Since \( \lim_{i \to \infty} a_i = 0 \), \( 0 \leq a_i \leq 1 \) for all but finitely many \( i \). Then \( 0 \leq a_i^n \leq a_i \) for all but finitely many \( i \). Thus \( \sum_i a_i^n \) converges by the comparison test. \( \square \)

**Definition 2.** If \( A \) is a Banach ring and \( r \) is a positive number, then \( A\langle r^{-1}T \rangle \) is defined to be the Banach ring

\[
A\langle r^{-1}T \rangle = \{ \sum_{i=0}^{\infty} a_i T^i \in A[[T]] : \sum_{i=0}^{\infty} ||a_i|| r^i < \infty \},
\]

with ring structure induced from \( A[[T]] \) and with norm

\[
|| \sum_i a_i T^i || = \sum_i ||a_i|| r^i.
\]

Note that \( A \) injects into \( A\langle r^{-1}T \rangle \) and that the norm on \( A\langle r^{-1}T \rangle \) extends the norm on \( A \). Berkovich uses \( A\langle r^{-1}T \rangle \) for what we denote as \( A\langle r^{-1}T \rangle \). We have been using \( A\{T_1, \ldots, T_n \} \) to denote the Tate algebra, which Berkovich denotes as \( A\{T_1, \ldots, T_n \} \).

---

\(^1\)Given any \( \epsilon > 0 \), \( ||a - b|| < \epsilon \) implies \( |\pi(a) - \pi(b)| \leq ||a - b|| < \epsilon \).
Lemma 5. $A(\langle r^{-1}T \rangle)$ is a Banach ring.

Proof. The triangle law is easy (and shows that $A(\langle r^{-1}T \rangle)$ is closed under addition). To check submultiplicativity, let $\sum_i a_i T^i$ and $\sum_i b_i T^i$ be two elements of $A(\langle r^{-1}T \rangle)$. Then

$$\left( \sum_i a_i T^i \right) \left( \sum_i b_i T^i \right) = \sum_i \left( \sum_{j=0}^i a_j b_{i-j} \right) T^i$$

and so

$$\| \left( \sum_i a_i T^i \right) \left( \sum_i b_i T^i \right) \| = \sum_i \| \left( \sum_{j=0}^i a_j b_{i-j} \right) \| r^i \leq \sum_i \sum_{j=0}^i (r^i \|a_j\|)(r^{i-j} \|b_{i-j}\|) \tag{4}$$

$$= \left( \sum_i \|a_i\| r^i \right) \left( \sum_i \|b_i\| r^i \right) \tag{5}$$

$$= \| \sum_i a_i T^i \| \cdot \| \sum_i b_i T^i \|. \tag{6}$$

This shows that $\| \cdot \|$ is submultiplicative, and additionally shows that $A(\langle r^{-1}T \rangle)$ is closed under multiplication. The equalities $\|0\| = 0$ and $\|1\| = 1$ follow because the norm on $A(\langle r^{-1}T \rangle)$ extends the norm on $A$. The fact that $A$ is closed under negation is easy, as is the verification that $\|a\| > 0$ for $0 \neq a \in A(\langle r^{-1}T \rangle)$.

The proof of completeness is similar to the proof that $L^p$ spaces are complete. We first show that if $a_j = \sum_i a_{i,j} T^i$ is a sequence of elements of $A(\langle r^{-1}T \rangle)$ such that $\sum_j \|a_j\| < \infty$, then $\sum_j a_j$ converges. For each $i$, $r^i \sum_j \|a_{i,j}\| \leq \sum_j \sum_i \|a_{i,j}\| r^i < \infty$. Therefore $\sum_j a_{i,j}$ converges to some value $b_i$. Then

$$\sum_i \|b_i\| r^i \leq \sum_i \left( \sum_j \|a_{i,j}\| \right) r^i < \infty,$$

so the formal power series $\sum_i b_i T^i$ is an element of $A(\langle r^{-1}T \rangle)$. Then for $j_0 > 0$

$$\| \sum_{j=0}^{j_0} a_i \| = \sum_i \left( b_i - \sum_{j=0}^{j_0} a_{i,j} \right) T^i$$

$$= \sum_i \|b_i - \sum_{j=0}^{j_0} a_{i,j}\| r^i = \sum_i \| \sum_{j>j_0} a_{i,j} \| r^i \leq \sum_j \sum_i \|a_{i,j}\| r^i = \sum_j \|a_j\|$$

which goes to 0 as $j_0 \to \infty$ because $\sum_j \|a_j\| < \infty$. So $\sum_j a_j = \beta$.

Now suppose that $\{a_j\}_{j=0}^\infty$ is an arbitrary Cauchy sequence in $A(\langle r^{-1}T \rangle)$. Choose $n_1 < n_2 < n_3 < \ldots$ so that $\|a_{n_j} - a_{n_{j+1}}\| \leq 2^{-j}$ for every $j$. Then $\sum_i (a_{n_j} - a_{n_{j+1}})$ converges by what we just showed. Therefore $\lim_{j \to \infty} a_{n_j}$ exists. But since the original sequence $\{a_j\}_{j=0}^\infty$ is a Cauchy sequence, it converges as long as any subsequence converges.
With all this out of the way, we can prove our main result:

**Theorem 1.** (Berkovich) Let \((A, || \cdot ||)\) be a nonzero Banach ring. Then the Berkovich spectrum \(\mathcal{M}(A)\) is nonempty.

**Proof.** Let \(M\) be a maximal ideal of \(A\). By Lemma 7, \(M\) is closed. By Lemma 8, this makes the field \(A/M\) a Banach ring with the residue seminorm. If we can show that \(\mathcal{M}(A/M)\) is empty, then there is some bounded multiplicative seminorm \(|| \cdot || : A/M \rightarrow \mathbb{R}_{\geq 0}\). Composing with the projection \(\pi : A \rightarrow A/M\) gives a bounded multiplicative seminorm on \(A\). Indeed, \(||\pi(a)|| \leq ||a||\) for \(a \in A/M\), so \(||\pi(a)|| \leq ||\pi(\alpha)|| \leq ||\alpha||\), making the composition bounded. The composition is a multiplicative seminorm because \(\pi\) is a homomorphism and \(|| \cdot ||\) is a multiplicative seminorm.

Replacing \(A\) with \(A/M\), we can therefore assume that \(A\) is a field, without loss of generality.

Let \(S\) be the set of bounded (but not necessarily multiplicative) seminorms on \(A\). The Banach norm \(|| \cdot ||\) on \(A\) is an element of \(S\), so \(S\) is nonempty. Assign a partial order on \(S\) by saying that \(|| \cdot ||_1 \leq || \cdot ||_2\) if \(||a||_1 \leq ||a||_2\) for every \(a \in A\). It is straightforward to check that Zorn’s lemma applies to \(S\), yielding a minimal element \(|| \cdot ||\). It remains to show that \(|| \cdot ||\) is multiplicative.

Note that since \(A\) is a field, \(|| \cdot ||\) must actually be a norm, since for nonzero \(a \in A\), \(1 = ||1|| = ||aa^{-1}|| \leq ||a|| ||a^{-1}||\), so \(||a|| \neq 0\). Also, if \(\{a_i\}_{i=0}^{\infty}\) is any Cauchy sequence with respect to \(|| \cdot ||\), then \(\{a_i\}_{i=0}^{\infty}\) is also a Cauchy sequence with respect to \(|| \cdot ||\) (the Banach norm), since \(|| \cdot ||\) is bounded by \(|| \cdot ||\). If \(\alpha = \lim_{n \rightarrow \infty} a_i\) with respect to the Banach norm \(|| \cdot ||\), then since \(||a_i - a_j|| \rightarrow 0\), \(\alpha\) is also the limit with respect to the norm \(|| \cdot ||\). So \(A\) is also a Banach ring with respect to \(|| \cdot ||\). Replacing \(|| \cdot ||\) with \(|| \cdot ||\), we can assume without loss of generality that \(|| \cdot || = || \cdot ||\). (Note that \(S\) only gets smaller when we decrease the Banach norm.)

At this point, we have reduced to the case where \(A\) is a field, and the Banach norm \(|| \cdot || = || \cdot ||\) is the unique bounded seminorm on \(A\). We will show that \(|| \cdot ||\) is multiplicative in two steps.

**Lemma 6.** The norm \(|| \cdot ||\) is power-multiplicative: \(||f^n|| = ||f||^n\) for all \(n \geq 1\).

**Proof.** Submultiplicativity already gives \(||f^n|| \leq ||f||^n\), so if power-multiplicativity fails, then \(||f^n|| < ||f||^n\) for some nonzero \(f \in A\) and some \(n > 1\). Let \(r = \sqrt[n]{||f||}\). Thus \(r^n = ||f^n|| < ||f||^n\) and \(r < ||f||\).

Consider the Banach ring \(A(\langle r^{-1}T \rangle)\) provided by Lemma 7. We claim that \(f - T\) is not invertible. Since \(A\) is a field, \(f^{-1}\) exists, and the inverse of \(f - T\) in \(A[[T]]\) is \(f^{-1}(1 - f^{-1}T)^{-1} = f^{-1}(1 + f^{-1}T + f^{-2}T^2 + \cdots)\).

If \(f - T\) had an inverse in \(A(\langle r^{-1}T \rangle)\), it would need to agree with this inverse in \(A[[T]]\), because \(A(\langle r^{-1}T \rangle)\) is a subring of \(A[[T]]\) and multiplicative inverses in commutative rings are unique, when they exist. But if \(f^{-1}(1 + f^{-1}T + f^{-2}T^2 + \cdots) \in A(\langle r^{-1}T \rangle)\), then certainly

\[
1 + f^{-1}T + f^{-2}T^2 + \cdots \in A(\langle r^{-1}T \rangle)
\]  

(7)

since \(A(\langle r^{-1}T \rangle)\) is closed under multiplication and contains \(f\). But by definition of \(A(\langle r^{-1}T \rangle)\), (7) implies that

\[
\sum_{i} |f^{-i}|r^i < \infty.
\]

By Lemma 7, this makes \(\sum_{i=0}^{\infty} |f|^n r^{in} < \infty\). However,

\[
\infty = \sum_{i=0}^{\infty} 1 = \sum_{i=0}^{\infty} r^{-in} r^{in} = \sum_{i=0}^{\infty} |f|^n |r^{-i} r^{in}| \leq \sum_{i=0}^{\infty} |f^{-i}| r^{in} < \infty,
\]

where the penultimate inequality follows by Lemma 7. This is a contradiction, and so \(f - T\) is not invertible.
Since $f - T$ is not invertible, $A \langle (r^{-1})T \rangle / (f - T)$ is nonzero, and has a residue seminorm $\cdot |_{\text{res}}$ by Lemma 7. Let $\phi : A \to A \langle (r^{-1})T \rangle / (f - T)$ be the composite of the embedding of $A$ into $A \langle (r^{-1})T \rangle$ and the canonical surjection. Define a seminorm $\cdot |_0$ on $A$ by pulling back the residue seminorm on $A \langle (r^{-1})T \rangle / (f - T)$:

$$|g|_0 = |\phi(g)|_{\text{res}}.$$ 

Then $\cdot |_0$ is a seminorm because $\phi$ is a homomorphism and $\cdot |_{\text{res}}$ is a seminorm. If $a \in A$, then $|a|_0 = |\phi(a)|_{\text{res}} \leq |a|$, since the norm on $A \langle (r^{-1})T \rangle$ extends the norm on $A$. Thus $\cdot |_0$ is a bounded seminorm. However, it isn’t the same as $\cdot |$ because

$$|f|_0 = |\phi(f)|_{\text{res}} = |\phi(T)|_{\text{res}} \leq ||T|| = r < |f|,$$

using the fact that $f$ and $T$ have the same image in the quotient $A \langle (r^{-1})T \rangle / (f - T)$. So $\cdot |_0$ is a strictly smaller seminorm than $\cdot |$, as an element of $\mathcal{S}$, contradicting the choice of $\cdot |$. □

Lemma 7. For every nonzero $f \in A$, $|f^{-1}| = |f|^{-1}$.

Proof. The proof is similar to the previous step. By submultiplicativity, $1 = |ff^{-1}| \leq |f|^{-1}||f|$, so $|f|^{-1} \leq |f|^{-1}$. The only way that $|f|^{-1} = |f|^{-1}$ can fail to hold is thus if $|f|^{-1} < |f|^{-1}$. In this case, let $r = |f|^{-1} - 1$, and consider the Banach ring $A \langle (r^{-1})T \rangle$ provided by Lemma 7. Again, we claim that $f - T$ is not a unit in $A \langle (r^{-1})T \rangle$. As before, invertibility of $f - T$ would imply that

$$\sum_i |f^{-1}|r^i < \infty.$$  

(8)

However, by the previous lemma, $|f^{-1}| = |f^{-1}|^i = r^{-i}$, so (8) says that

$$\infty = \sum_i 1 = \sum_i r^{-i}r^i = \sum_i |f^{-1}|r^i < \infty,$$

a contradiction. Thus $f - T$ is not invertible, as before.

Then consider the homomorphism $\phi : A \to A \langle (r^{-1})T \rangle / (f - T)$, and the pullback $\cdot |_0$ of the residue seminorm $\cdot |_{\text{res}}$ on $A \langle (r^{-1})T \rangle / (f - T)$ as in the previous lemma. As before, $\cdot |_0$ is a bounded seminorm on $A$, but

$$|f|_0 = |\phi(f)|_{\text{res}} = |\phi(T)|_{\text{res}} \leq ||T|| = r = |f|^{-1} - 1 < |f|.$$ 

The last step follows by inverting $|f|^{-1} > |f|^{-1}$. Thus $\cdot |_0$ is strictly smaller than $\cdot |$ (as an element of $\mathcal{S}$), a contradiction. □

Multiplicativity of $\cdot |$ then follows easily from the previous lemma. If $f, g \in A$, then the lemma and submultiplicativity yield

$$|fg|^{-1} = |f^{-1}g^{-1}| \leq |f|^{-1}|g|^{-1} = |f|^{-1}|g|^{-1} \leq |fg|^{-1}$$

and so $|f||g| = |fg|$, unless $f$ or $g$ vanishes, in which case multiplicativity is obvious. □

This theorem has several corollaries. Here is one:

Corollary 1. If $A$ is a nonzero Banach ring, then $f \in A$ is invertible if and only if $f(x) \neq 0$ for all $x \in \mathcal{M}(A)$, where $f(x)$ is the image of $f$ in $\mathcal{H}(x) = k(x)$ under the canonical map $A \to k(x) \to k(x)$ induced by $x$. 
Proof. If $f$ is invertible, then letting $g$ be an inverse, we have $f(x)g(x) = 1(x) = 1$ for all $x \in \mathcal{M}(A)$. Therefore, $f(x) \neq 0$ for all $x \in \mathcal{M}(A)$. Conversely, suppose that $f$ is not invertible. Then $f \in M$ for some maximal ideal $M$. By Lemmas ?? and ??, the residue seminorm on $A/M$ is a norm making $A/M$ into a Banach ring. Thus by the theorem, $\mathcal{M}(A/M)$ is nonempty. Taking any multiplicative seminorm $|\cdot|: A/M \to \mathbb{R}_{\geq 0}$, compose with the map $A \to A/M$ and get a point $x \in \mathcal{M}(A)$ such that $|f|_x = 0$. Thus $f \in \ker(A \to k(x)) = \ker(A \to \mathcal{H}(x))$, by definition of $k(x)$. So $f(x) = 0$.  

In the next class, we will see additional corollaries of Theorem ??.
1 Spectral radii and Gel’fand transform

Let $\mathcal{A}$ be a nonzero Banach ring. Last time, we showed that $\mathcal{M}(\mathcal{A}) \neq \emptyset$.

Definition 1. For $f \in \mathcal{A}$, define the spectral radius of $f$ to be

$$\rho(f) := \inf_n \sqrt[n]{\|f\|} = \lim_{n \to \infty} \sqrt[n]{\|f\|}.$$  

Theorem 1. For all $f \in A$,

$$\rho(f) = \max_{x \in \mathcal{M}(\mathcal{A})} |f(x)|.$$  

Remarks 1. (i) Recall that $f(x) \in \mathcal{H}(x)$, so it make sense to talk about its norm $|f(x)|$. Also, $\mathcal{M}(\mathcal{A})$ is compact, so $|f(x)|$ achieves a maximum value on $\mathcal{M}(\mathcal{A})$.

(ii) One can think of this result as a maximum modulus principle for Banach rings.

(iii) Consider the map

$$\mathcal{A} \to \prod_{x \in \mathcal{M}(\mathcal{A})} \mathcal{H}(x)$$

defined by

$$f \mapsto \hat{f} := (f(x))_{x \in \mathcal{M}(\mathcal{A})}.$$  

This map is called the Gel’fand transform of $f$. Here, the product is not the product in the category of rings. Instead, if $(\mathcal{B}_i, \|\cdot\|_i)_{i \in I}$ is a family of Banach ring, set

$$\mathcal{B} = \prod_{i \in I} \mathcal{B}_i$$

to be a Banach rings whose elements are of the form $f = (f_i)$ such that there exists $c > 0$ with $\|f_i\|_i \leq c$ for all $i$. The norm is defined by

$$\|f\| := \sup_i \|f_i\|.$$  

By definition, the Gelfand transform is continuous. Moreover, the theorem says that

$$\rho(f) = \|\hat{f}\|.$$  

In particular, the kernel of the Gelfand transform is the kernel of $\rho$. Note also that if $\|\cdot\|$ is power multiplicative, then $\rho(f) = \|f\| = \|\hat{f}\|$.

Exercise 1. Prove that $\rho$ is a bounded power-multiplicative seminorm.
Proof of Theorem 1. Write
\[ |f|_{\text{sup}} = \max_{x \in \mathfrak{a}} |f(x)|. \]

Then
\[ |f^n|_{\text{sup}} = |f^n|_{\text{sup}} \leq \|f^n\|, \]
so \( |f|_{\text{sup}} \leq \rho(f) \). Now, suppose that there exists \( f \in \mathfrak{a} \) such that \( |f|_{\text{sup}} < \rho(f) \). Choose a real number \( r \) such that
\[ |f|_{\text{sup}} < r < \rho(f). \]
Consider the Banach ring \( \mathcal{B} = \mathfrak{a}((rT)) \). We have \( \|T\| = r^{-1} \), so for all \( x \in \mathcal{M}(\mathcal{B}), |T(x)| \leq r^{-1}. \) Therefore,
\[ |(fT)(x)| < r \cdot r^{-1} = 1 \]
so \( (1 - fT)(x) \neq 0 \) for all \( x \in \mathcal{M}(\mathcal{B}). \) Then by a result in the last lecture, \( 1 - fT \) is invertible in \( \mathcal{B} \) with inverse
\[ \sum_{i=0}^{\infty} f^iT^i. \]

Then
\[ \sum_{i=0}^{\infty} \|f^i\|r^{-1} < \infty, \]
so by the root test, \( \rho(f) \leq r \), which is a contradiction. \( \square \)

2 Common extensions of two fields

Consider a diagram
\[
\begin{array}{ccc}
K_1 & \longrightarrow & k \\
\downarrow & & \\
K_2 & \longrightarrow & k
\end{array}
\]
of fields. Does there exist a field \( K \) extending this diagram to a diagram
\[
\begin{array}{ccc}
K & \longrightarrow & K_1 \\
\downarrow & & \downarrow \\
K_2 & \longrightarrow & k
\end{array}
\]
which is commutative? One cannot answer the question using the compositum operation because the compositum of \( K_1 \) and \( K_2 \) is only defined once we have constructed a common overfield \( K \)."

Step 1. \( A := K_1 \otimes_k K_2 \) is a nonzero \( k \)-algebra.
This is true because if \( V_1, V_2 \) are nonzero \( k \)-vector spaces, then \( V_1 \otimes_k V_2 \) is also nonzero.

Step 2. By step 1, we have \( \text{Spec}(A) \neq \emptyset \), so there exists a prime ideal \( p \in A \), and we have natural maps
\[ K_i \hookrightarrow \text{Frac}(A/p). \]
Set \( K = \text{Frac}(A/p). \)

Fact. (This will not be used in this course.) Every compositum of \( K_1, K_2 \) over \( k \) can be obtained in this way, and different prime ideals \( p \) yield non isomorphic \( K \)'s, i.e., \( \text{Spec}(K_1 \otimes K_2) \) classifies the classes of compositum.
**Theorem 2.** Given complete valued fields $K_1$ and $K_2$ over a field $k$, there exists a complete valued field $K$ extending $K_1$ and $K_2$.

**Lemma 1.** Any bounded map of valued fields is an isometric embedding.

**Proof.** Let $\varphi : k \to K$ be the map. If $|a| < |\varphi(a)|$ for some $a \in k$, then

$$\frac{|\varphi(a^n)|}{|a^n|} \to \infty$$

as $n \to \infty$, contradicting boundedness. We get a similar contradiction if $|a^{-1}| < \varphi(a^{-1})$, i.e., if $|a| > |\varphi(a)|$.

We used the tensor product $K_1 \otimes_k K_2$ to construct the common extension $K$. However, this is in general not a valued field. We need instead the completed tensor product. We have a norm on $K_1 \otimes_k K_2$ defined by

$$|z| = \inf \left\{ \sum_{i=1}^{r} \| x_i \| \| y_i \| : z = \sum_{i=1}^{r} x_i \otimes y_i, \text{ all writings} \right\}.$$

If $K_1, K_2, k$ are non-archimedean fields, we use a different norm defined by

$$|z| = \inf \left\{ \inf_{1 \leq i \leq r} \| x_i \| \| y_i \| : z = \sum_{i=1}^{r} x_i \otimes y_i, \text{ all writings} \right\}.$$

Set $K_1 \hat{\otimes}_k K_2$ to be the completion of $K_1 \otimes_k K_2$ respect to one of these norm. This $K_1 \hat{\otimes}_k K_2$ is called the completed tensor product.

**Lemma 2.** If $K_1, K_2$ are complete valued field extensions of $k$, their completed tensor product $K_1 \hat{\otimes}_k K_2$ is nonzero.

**Proof.** We can prove that $\| 1 \otimes 1 \| = 1$. This is an exercise, or one can see ([BGR],4.19). More generally, we have a canonical injection [Gruson]

$$K_1 \otimes_k K_2 \hookrightarrow K_1 \hat{\otimes}_k K_2,$$

but we will not use this fact.

**Proof of Theorem 2.** Choose a point $x \in M(K_1 \hat{\otimes}_k K_2)$. Recall that $\mathcal{H}(x)$ is the complete residue field of $x$. The natural injections $K_i \hookrightarrow \mathcal{H}(x)$ are bounded. By Lemma 1, they are isometric embeddings. Now, set $K = \mathcal{H}(x)$.

## 3 One more example of $\mathcal{M}(\mathcal{A})$

Consider $\prod_{i \in I} K_i$ where $K_i$ are complete valued fields. Then

$$\mathcal{M}(K_i) = \{ \text{pt} \}$$

for all $i$. This is an exercise. Also,

$$M \left( \prod_{i \in I} K_i \right)$$

is homeomorphic to the Stone-Čech compactification $SC(I) := \{ \text{ultrafilters on I} \}$ of $I$. In $SC(I)$, subsets $\{ \mathcal{F} : J \in \mathcal{F} \}$, $J \subset I$ are open and generate the topology. For the detail, see ([Berk],1.2.3).
This lecture discusses the classification of points and the structure sheaf on $\mathcal{M}(\mathbb{Z})$. Also, this lecture briefly introduces rigid analytic geometry.

1 $\mathcal{M}(\mathbb{Z})$

Recall that the Berkovich spectrum $\mathcal{M}(\mathcal{A})$ is the set of all nonzero bounded multiplicative seminorms on $\mathcal{A}$, with the weakest topology such that $| \cdot | \mapsto |f|$ is continuous for all $f \in \mathcal{A}$. First we recall Ostrowski’s theorem.

**Theorem 1.** (Ostrowski) Every nonzero absolute value on $\mathbb{Q}$ is equivalent to the usual absolute value $| \cdot |_{\infty}$ or the absolute value $| \cdot |_p$, where $p$ is a prime number, defined as follows:

$$
|0| = 0,
|p^k \frac{a}{b}|_p = p^{-k}, \text{where } k \in \mathbb{Z}, a, b \in \mathbb{Z}\setminus\{0\}, \gcd(a, p) = \gcd(b, p) = 1.
$$

The following theorem classifies the points in $\mathcal{M}(\mathbb{Z})$.

**Theorem 2.** Every point in $\mathcal{M}(\mathbb{Z})$ is of one of the following 4 types:

- **(Z1)** The $p$-trivial seminorm $| \cdot |_{p, \infty}$, where $p$ is a prime number:
  $$
  |n|_{p, \infty} = \begin{cases} 
  0, & \text{if } p|n, \\
  1, & \text{otherwise}.
  \end{cases}
  $$

- **(Z2)** The trivial seminorm $| \cdot |_0$:
  $$
  |n|_0 = \begin{cases} 
  0, & \text{if } n = 0, \\
  1, & \text{otherwise}.
  \end{cases}
  $$

- **(Z3)** The seminorm $| \cdot |_{p, \epsilon}$, where $p$ is a prime number and $\epsilon \in (0, +\infty)$:
  $$
  |n|_{p, \epsilon} = |n|_p^{\epsilon}.
  $$

- **(Z4)** The seminorm $| \cdot |_{\infty, \epsilon}$, where $p$ is a prime number and $\epsilon \in (0, 1]$:
  $$
  |n|_{\infty, \epsilon} = |n|_{\infty}^{\epsilon}.
  $$

*Proof.* Exercise. \hfill \Box
Figure 1: A picture of $\mathcal{M}(\mathbb{Z})$. Taken from Poineau, An introduction to the Berkovich line over $\mathbb{Z}$.

Figure 1 illustrates the topology on $\mathcal{M}(\mathbb{Z})$. Each branch is homeomorphic to a real interval. The open neighborhoods $U$ of the root $|\cdot|_0$ are such that

- The intersection of $U$ with each branch is open.
- $U$ contains all but finitely many branches.

It can be shown (exercise) that this space is compact and Hausdorff.

**Remark 1.** The topology of $\mathcal{A}^1_{\text{Berk},k}$ with $k$ trivially valued is similar to that of $\mathcal{M}(\mathbb{Z})$, except that the top point is missing.

**Definition 1.** For $a \in \mathbb{Z}$ and $x \in \mathcal{M}(\mathbb{Z})$, let $a(x)$ denote the image of $a$ under the canonical map $\mathbb{Z} \to \mathcal{H}(x)$. A rational function without poles on an open set $U \subset \mathcal{M}(\mathbb{Z})$ is a map

$$f : U \to \prod_{x \in U} \mathcal{H}(x)$$

of the form $f(x) = a(x)/b(x)$, where $a, b \in \mathbb{Z}$ and $b(x)$ is nonzero on $U$.

**Definition 2.** Let $U$ be an open subset of $\mathcal{M}(\mathbb{Z})$. An analytic function on $U$ is a map

$$f : U \to \prod_{x \in U} \mathcal{H}(x)$$
which is a locally uniform limit of elements of $\mathbb{Q}$ without poles, i.e. for any $x \in U$, there exists an open neighborhood $U'$ of $x$ in $U$ such that for any $\epsilon > 0$, there exists a rational function $g$ without poles on $U$ such that $|f(x') - g(x')| < \epsilon$ for all $x' \in U'$.

The analytic functions form the structure sheaf on $\mathcal{M}(\mathbb{Z})$.

**Remark 2.** We can get the definition of the structure sheaf on $\mathbb{A}^1_{\text{Berk}, k}$ by replacing $\mathbb{Z}$ by $k[T]$ and replacing $\mathbb{Q}$ by $k(T)$. This works more generally for $\mathbb{A}^n_{\text{Berk}, k}$. It follows from the Gelfand–Mazur theorem that $\mathbb{A}^n_{\text{Berk}, \mathbb{C}}$ is homeomorphic to $\mathbb{C}^n$ with the usual topology. Runge proved that up to this homeomorphism, the analytic functions on an open subset $U$ of $\mathbb{A}^n_{\text{Berk}, \mathbb{C}}$ is the same as the usual analytic functions on $U$. Thus, $\mathbb{A}^n_{\text{Berk}, \mathbb{C}}$ and $\mathbb{C}^n$ are isomorphic as locally ringed spaces.

Figure 2: A picture of the structure sheaf on $\mathcal{M}(\mathbb{Z})$. Taken from Poineau, An introduction to the Berkovich line over $\mathbb{Z}$.

Figure 2 illustrates the structure sheaf on $\mathcal{M}(\mathbb{Z})$.

Let $U = \mathcal{M}(\mathbb{Z}) - \{|\cdot|_0\}$. Let $j: U \to \mathcal{M}(\mathbb{Z})$ be the canonical embedding. Berkovich observes that $(j_* \mathcal{O}_U)_{|0}$ is the ring of adeles of $\mathbb{Q}$, and $(j_* \mathcal{O}_U)_{|0}$ is the group of ideles of $\mathbb{Q}$. This also works if $\mathbb{Z}$ is replaced by the ring of integers of a number field.

## 2 A crash course in rigid analytic geometry

Let $k$ be a complete non-trivially valued non-archimedean field. An affinoid algebra over $k$ is $\mathcal{A} = k\langle T_1, \ldots, T_n \rangle / I$ for some (closed) ideal $I$ of $k\langle T_1, \ldots, T_n \rangle$, with the residue norm. It follows that $\mathcal{A}$ is a
Banach algebra over $k$. If $\mathcal{A}$ is isomorphic to $k\langle T_1, \ldots, T_m \rangle / J$ for some ideal $J$ as well, the resulting residue norms on $\mathcal{A}$ may be different. However, it can be shown that they are always equivalent.

Fact: for every maximal ideal $m$ of $\mathcal{A}$, $\mathcal{A}/m$ is a finite extension of $k$. In particular, there is a canonical extension of the absolute value on $k$ to an absolute value on $\mathcal{H}(x) = k(x) := \mathcal{A}/m_x$, where $m_x$ is a maximal ideal of $\mathcal{A}$ and $x$ is the corresponding point in $\text{Sp}(\mathcal{A}) := \{\text{maximal ideals of } \mathcal{A}\}$. The canonical topology on $\text{Sp}(\mathcal{A})$ is the weakest topology such that $x \mapsto |f(x)|$ is continuous for all $f \in \mathcal{A}$, where $f(x)$ is the image of $f$ in $k(x)$. $\text{Sp}(\mathcal{A})$ is a totally disconnected, non–locally compact topological space.

**Example 1.** If $k$ is algebraically closed and $\mathcal{A} = k\langle T \rangle$, then $\text{Sp}(\mathcal{A})$ with its canonical topology is homeomorphic to $k^0$ with its usual topology.

Now we define the *special $G$–topology* on $\text{Sp}(\mathcal{A})$. A rational subdomain of $\text{Sp}(\mathcal{A})$ is $V = \{x \in \text{Sp}(\mathcal{A}) : |f_i(x)| \leq |g(x)|\}$ for some $f_1, \ldots, f_r, g \in \mathcal{A}$ having no common zeros. A special subdomain is a finite union of rational subdomains. The special $G$–topology on $\text{Sp}(\mathcal{A})$ is the following data:

- Admissible opens: special subdomains.
- Admissible coverings: finite coverings by special subdomains.

Let $V$ be a rational domain. Define

$$\mathcal{A}_V := \mathcal{A}(x_1, \ldots, x_r)/(gx_1 - f_1, \ldots, gx_r - f_r).$$

In fact, $\mathcal{A}_V$ is independent of the presentation. Moreover, $\text{Sp}(\mathcal{A}_V)$ is homeomorphic to $V$.

**Theorem 3.** (Tate) The association $V \mapsto \mathcal{A}_V$ extends to a sheaf on the special $G$–topology. In particular, if $\{V_i\}$ is a finite covering of $V$ by special subdomains, then the sequence

$$0 \to \mathcal{A}_V \to \prod_i \mathcal{A}_{V_i} \to \prod_{i,j} \mathcal{A}_{V_i \cap V_j}$$

is exact.
Let us begin by recalling the following two definitions:

**Definition 0.1.** A lattice is a partially ordered set \((L, \leq)\) which has pairwise greatest lower bounds and least upper bounds, denoted \(\land\) and \(\lor\) respectively. If \(\land\) and \(\lor\) distribute over each other, we say that \(L\) is distributive. If there is some \(0 \in L\) with \(0 \leq x\) for all \(x \in L\), then we say that \(L\) is lower-bounded.

**Definition 0.2.** We write \(x \triangleleft x'\)—said "\(x\) is inner in \(x'\)"—if, for all \(z \geq x'\), there exists \(w\) with \(x \land w = 0\), \(x' \lor w = z\). If \(L\) is a lower-bounded distributive lattice with the property that, for all \(x, y \in L\) with \(x \land y = 0\), there exists \(x \triangleleft x'\) with \(x' \lor y = 0\), then we say that \(L\) is an overconvergent lattice.

Let \(k\) be a complete, nontrivially valued non-archimedean field, let \(A\) be a (strict) affinoid algebra over \(k\), and let \(X = \text{Sp}(A)\). Let \(L\) be the lattice of special subdomains of \(X\), so \(L\) is generated by rational domains \(R(f_0; f_1, \ldots, f_s) = \{x \in X : |f_i(x)| \leq |f_0(x)|, i = 1, \ldots, s\}\), where \(f_0, \ldots, f_s\) generate the unit ideal in \(A\).

Note that the rational domains form a basis for "clopen" sets for the canonical topology on \(X\). If \(X^{\text{an}} = \mathcal{M}(A)\) is the Berkovich spectrum of \(X\), then rational domains form a basis of compact neighborhoods of \(X^{\text{an}}\) (this follow from the definition of the Berkovich topology). We'll recall the construction of \(X^{\text{an}}\) from the data of \(X\) and the lattice \(L\).

**Theorem 0.3.** The lattice \(L\) is overconvergent.

For a thorough discussion when \(X\) is an affinoid, see Chapter 7 of *Rigid Analytic Geometry and Its Applications* by Fresnel and van der Put. The following lemma is the key point in the proof of the theorem above:

**Lemma 0.4.** Suppose \(R = R(f_0; f_1, \ldots, f_s)\) and \(R' = R(f'_0; f'_1, \ldots, f'_s)\) are disjoint rational domains in \(X\). Then there exists \(\rho \in \sqrt{|k^*|}\) with \(\rho > 1\) such that \(R_\rho = R(\rho f_0; f_1, \ldots, f_s)\) and \(R'\) are still disjoint in \(X\).

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Note that since $k$ is nontrivially valued, $\sqrt{|k^*|}$ is dense in $\mathbb{R}$. The proof of this lemma is left as an exercise, noting the maximum modulus principle (see BGR) and that $R$ is inner in $R_p$.

**Theorem 0.5.** (Dudzik) Suppose $X$ is a Hausdorff topological space and $L$ is an overconvergent lattice of subsets of $X$ that form a neighborhood base. Then:

1. There is a canonical surjective map $\mathcal{P}(L) \to \mathcal{M}(L)$ sending a prime filter $P$ to the unique maximal filter containing it. Recall that the topology on the set $\mathcal{P}(L)$ of prime filters is generated by the open sets $\{P : V \in P\}$ for $V \in L$. Hence we can equip $\mathcal{M}(L)$ with the quotient topology.
2. With the quotient topology, $\mathcal{M}(L)$ is the unique locally compact Hausdorff space containing $X$ as a dense subspace such that $\{\overline{V} : V \in L\}$ forms a basis of compact neighborhoods in $\mathcal{M}(L)$.

**Corollary 0.6.** Let $X = \text{Sp}(A)$ as above. Then we have a homeomorphism $\mathcal{M}(L) \cong X^\text{an}$. 

The corollary follows from the fact that $\text{Sp}(A)$ is dense in $\mathcal{M}(A)$. We will now sketch the proof of this fact:

**Lemma 0.7.** Let $k$ and $A$ be as above. Then the natural map $\text{Sp}(A) \to \mathcal{M}(A)$ has a dense image.

**Proof.** (Sketch) WLOG, let us assume that $|k^*|$ is dense. Then any point $x_0 \in \mathcal{M}(A)$ has a basis of open neighborhoods of the form

$$U = \{x \in \mathcal{M}(A) : |f_i(x)| < 1, |g_j(x)| > 1\}$$
for $i = 1, \ldots, m$, $j = 1, \ldots, n$, and $f_i, g_j \in A$. Choose $p_i, q_j \in k$ such that $|f_i(x_0)| < |p_i| < 1$ and $|g_j(x_0)| > |q_j| > 1$. Let $B = A(T_i, S_j)/(p_iT_i - f_i, g_jS_j - q_j)$. Then we have

$$\mathcal{M}(B) \cong \{x \in \mathcal{M}(A) : |f_i(x)| \leq |p_i| \text{ and } |g_j(x)| \geq |q_j|\}.$$ 

In particular, we have $\mathcal{M}(B) \subseteq U$. Now $B \neq 0$ implies that there exists $M \in Sp(B)$ (it is an exercise to show this). The image of the canonical map

$$Sp(B) \rightarrow Sp(A) \rightarrow \mathcal{M}(A)$$

lies in $U$ and contains the point corresponding to $M$. \qed

In the next exercise, we will make the homeomorphism $\mathcal{M}(L) \simeq X^an$ more explicit by giving a correspondence between between maximal filters and points of $X^an$.

**Exercise:** Let $k$ and $A$ be as above. Then there is a bijection $\mathcal{M}(A) \leftrightarrow \mathcal{M}(L)$ sending $F \in \mathcal{M}(L)$ to the element $\cdot |_F$, where for $f \in A$, we let $|f|_F = \inf_{V \in F} |f|_V$. Note that if $V$ is a rational subdomain of $X$, then $|f|_V = \sup_{x \in V} |f(x)|$, or if $\pi : A \rightarrow A_V = B$ is the natural map then $|f|_V = ||\pi(f)||$. In the other direction, we send $\cdot |_x \in \mathcal{M}(A)$ to the maximal filter $F_x = \{V \in L : V \text{ contains a rational domain } R = R(f_0; f_1, \ldots, f_s) \text{ such that } |f_i|_x \leq |f_0|_x \forall i\}$.

Let’s illustrate this correspondence in the case of the unit disc.

**Example 0.8.** Let $k$ be algebraically closed, and let $X = \text{Sp}(k\langle T \rangle) \simeq B^1 = k^0$. Let $X^an = \mathcal{M}(k\langle T \rangle)$.

Type 1 points: $x \leftrightarrow a \in k$ corresponds to $F = \{V : a \in V\}$.

Type 2 and 3 points: $x \leftrightarrow D = D(a, r)$ corresponds to $F = \{V : V \supseteq D' \text{ for some } D' = D\setminus \cup_{i=1}^s D(a_i, r_i)\}$ where $D(a_i, r_i)$ denotes the open disc. Note that in type 3, $D'$ won’t itself be a rational domain.

Type 4 points: $x \leftrightarrow D_1 \supseteq D_2 \supseteq \cdots$ with $\cap D_n \neq \emptyset$. Then $x$ corresponds to $F = \{V : V \supseteq D_n \text{ for some } n\}$. 
Today, we will address the question “where are the type V points?” Throughout, \( k = \bar{k} \) is complete with respect to a nonarchimedean valuation. Our running example shall be \( X = \mathbb{B}^1 = \text{Sp}(k(T)) \), equipped with the special G-topology: admissible opens are special subsets and admissible covers are covers with a finite refinement by special sets. A *prime G-filter* is a collection \( \mathcal{F} \) of special subsets such that:

- \( \emptyset \notin \mathcal{F} \)
- If \( U, V \in \mathcal{F} \), then \( U \cap V \in \mathcal{F} \)
- If \( U \in \mathcal{F} \) with \( U' \supseteq U \), then \( U' \in \mathcal{F} \)
- If \( V_1 \cup \cdots \cup V_n \in \mathcal{F} \), then some \( V_i \in \mathcal{F} \)

In our example \( X = \mathbb{B}^1 \), the special sets are just finite unions of affinoid domains of the form \( D \setminus \bigcup D_i^- \), where \( D \subset D(0,1) \) is a closed disc and \( D_i^- \subset D \) are open discs.

It is an exercise to show that all rational domains have the above form. Then by the Gerritzen-Grauert theorem, all affinoid domains have this form as well.

Here is an example of a type V point, i.e. a prime filter that is not maximal. Let \( a \in k^0 \), i.e. \( |a| \leq 1 \). Then let us define

\[
\mathcal{F} := \{ V : V \supseteq \{ z \in \mathbb{B}^1 : r' < |z - a| < 1 \} \text{ for some } 0 < r' < 1 \}
\]
We have that $\mathcal{F}$ is a prime filter (it is an exercise to prove this). Furthermore, $\mathcal{F}$ is properly contained in the type II maximal filter associated to the Gauss point, namely

$$\mathcal{F}_\text{Gauss} := \{V : V \supseteq D(0,1) \setminus \bigcup D(a_i, r_i)^- \text{ for some } a_i, r_i\}.$$  

For example, $V = D(0,1) \setminus D(0,1)^- \in \mathcal{F}_\text{Gauss}$ but $V$ is not in the prime filter $\mathcal{F}$ above. Therefore, $\mathcal{F}_\text{Gauss}$ is the unique maximal filter containing $\mathcal{F}$.

Similarly, for each open disc $D(a,1)^- \subset D(0,1)$, we get an associated “type V” point of the Huber adic space $\mathcal{P}(X)$; each such point is associated to the Gauss point.

We remark that the tangent directions at $\zeta_\text{Gauss}$ are in 1 to 1 correspondence with closed points of $\mathbb{A}^1_{\bar{k}}$, which are in 1−1 correspondence with open discs $D(a,r)^- \subset D(0,1)$. For more general points $\zeta = \zeta_{a,r}$ of type II corresponding to $D(a,r)$, we have

$$\{\text{tangent directions at } \zeta\} \leftrightarrow \{\text{closed points of } \mathbb{P}^1_{\bar{k}}\}$$

$$\leftrightarrow \{\text{open discs } D(a',r)^- \subset D(a,r)\} \cup \{D(0,1) \setminus D(a,r)\}.$$  

For each such $\zeta$ of type II and each tangent direction at $\zeta$, we obtain an associated type V point of Huber adic space.

What is the topology on the space $\mathcal{P}(X)$ of prime filters? Again, let $X = \mathbb{B}^1$, $k = \bar{k}$.

The point $\zeta_\text{Gauss}$ is closed in $\mathcal{M}(X)$, whereas in $X^{\text{ad}} = \mathcal{P}(X)$, its closure is $\zeta_\text{Gauss} \cup \{\text{associated type V points}\}$. Recall that $\mathcal{P}(X)$ is quasicompact but not Hausdorff. The map $\rho : \mathcal{P}(X) \to \mathcal{M}(X)$ is continuous, with a
natural section $\mathcal{M}(X) \to \mathcal{P}(X)$. In fact, $\mathcal{M}(X)$ is the maximal Hausdorff quotient of $\mathcal{P}(X)$. Furthermore, we have a homeomorphism

$$\rho^{-1}(\zeta_{\text{Gauss}}} \in (\mathbb{B}^1)^{an}) \simeq \mathbb{A}^1_k$$

considered as schemes.
Prime filters and semivaluations

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Let $X = \text{Sp}(A)$, where $A$ is an affinoid algebra over $k$, a complete non-trivially valued non-archimedean field. In this lecture, we aim to show the equivalence between the set of prime filters $\mathcal{P}(X)$ on the special $G$-topology of $X$ and Huber’s adic space of continuous semivaluations on $A$. We begin with a few definitions.

**Definition 1.** Let $K$ be a field. A valuation ring of $K$ is an integral domain $R$ with field of fractions $K$ such that for all $x \in K$, either $x \in R$ or $x^{-1} \in R$. The height or rank of a valuation ring $R$ is its Krull dimension, and is denoted by $\text{ht}(R)$.

**Remark.** The ring of integral elements of $k$ is a valuation ring of height 1.

**Definition 2.** Let $(A, || \cdot ||)$ be a Banach algebra over $k$ with $|| \cdot ||$ extending the norm on $k$. Choose an element $\pi \in k^0$ with $0 < ||\pi|| < 1$. A semivaluation on $A$ is a pair $(p, B)$, where $p \in \text{Spec}(A)$ and $B$ is a valuation ring of Frac($A/p$). A semivaluation $(p, B)$ is **continuous** if the following conditions hold:

1. The image of $A^0$ in Frac($A/p$) is contained in $B$.
2. $\bigcap_{n=1}^{\infty} \pi^n B = 0$.

It is easily checked that the notion of continuity is independent of the choice of $\pi$. Additionally, a continuous semivaluation $(p, B)$ with $B$ a height 1 valuation ring is equivalent to giving a bounded multiplicative seminorm on $A$. Indeed, the valuation ring $B$ gives rise to an equivalence class of norms on Frac($A/p$). For $B$ with $\text{ht}(B) = 1$, condition (2) above is equivalent to $\pi$ lying in the maximal ideal of $B$, so there is a unique norm $|| \cdot ||_B$ in this equivalence class satisfying $||\pi||_B = ||\pi||$. Then we define $|| \cdot ||_{(p, B)}$ by setting $|a|_{(p, B)} = |\bar{a}|_B$. Clearly, this is a multiplicative seminorm on $A$, so it remains to show that $|| \cdot ||_{(p, B)}$ is bounded.

It is an easy exercise to show that the boundedness of $|| \cdot ||_{(p, B)}$ is equivalent to showing that the projection map $(A, || \cdot ||) \to (A/p, || \cdot ||_B)$ is continuous. Let $B(\bar{a}, r)$ be an open ball of radius $r > 0$ about $\bar{a} \in A/p$. We may assume $r < 1$. Additionally, since the map $A/p \to A/p$ given by $\bar{x} \mapsto \bar{x} - \bar{a}$ is a homeomorphism, we may assume $\bar{a} = 0$. Thus, we are reduced to showing that there is some $r' > 0$ such that $||a|| < r'$ implies $|\bar{a}|_B < r$. Choose $n$ such that $||\pi^n||_B < r$. Then we can take $r' = ||\pi^n||_B$. Indeed, if $||a|| < r'$, then

$$||\pi^{-n} a|| \leq ||\pi^{-n}|| \cdot ||a|| = ||\pi^{-n}||_B \cdot ||a|| = ||\pi^n||^{-1}_B \cdot ||a|| = (r')^{-1} ||a|| < 1.$$ 

This shows that $\pi^{-n} a \in A^0$, and so by (1) we have $\bar{\pi}^{-n} a \in B$. It follows that $||\pi^{-n} \bar{a}||_B \leq 1$, in other words $|\bar{a}|_B \leq ||\pi|| = r' < r$.

On the other hand, if $|| \cdot ||$ is a bounded multiplicative seminorm on $A$, we can construct a height 1 continuous semivaluation by setting $p := \ker(|| \cdot ||)$ and $B$ to be the valuation ring of Frac($A/p$) determined by $|| \cdot ||$. Since $|| \cdot ||$ is bounded, for $a \in A^0$ we have $|\bar{a}| \leq ||a||$, and so $\bar{a}$ is power bounded under $|| \cdot ||$. But the set of power bounded elements of $A/p$ under $|| \cdot ||$ is precisely $B$, so (1) holds. Also, $||\pi|| \leq ||\pi|| < 1$, which is equivalent to (2) as stated above. Therefore, $(p, B)$ is continuous. It is clear that these two constructions are inverses of each other, and so we have established a bijection

$$\{\text{height 1 continuous semivaluations}\} \leftrightarrow \{\text{bounded multiplicative seminorms}\}.$$
We wish to generalize this bijection to the set of all continuous semivaluations on $A$, which we will denote by $\text{Val}(A)$. Previously, we found a bijection between bounded multiplicative seminorms and the set of maximal filters $\mathcal{M}(X)$ on $X$ when $A$ is an affinoid $k$-algebra, so a natural extension to consider is a bijection between $\text{Val}(A)$ and the set of prime filters $\mathcal{P}(X)$, again when $A$ is affinoid. First, given $P \in \mathcal{P}(X)$, we can construct a continuous semivaluation as follows. Define $\mathcal{O}_P := \lim_{V \downarrow V} A_V$, where the limit is taken over all $V \in P$ corresponding to rational subdomains of $A$. We can put a seminorm on $\mathcal{O}_P$ by setting $\|f\|_P := \inf_{V \in P} \|f\|_V$. Finally, set $\mathcal{O}_P^0 := \lim_{V \downarrow V} A_V^0$.

**Lemma 1.**

1. $\mathcal{O}_P$ is a local ring with unique maximal ideal $\mathcal{M}_P = \{ f \in \mathcal{O}_P : \|f\|_P = 0 \} \subseteq \mathcal{O}_P^0$.

2. $k_P^0 := \mathcal{O}_P^0 / \mathcal{M}_P$ is a valuation ring of $k_P := \mathcal{O}_P / \mathcal{M}_P$.

3. Let $p = \ker(A \to k_P)$, $\psi : \text{Frac}(A/p) \to k_P$, and $B := \psi^{-1}(k_P^0)$. Then $(p, B)$ is a continuous semivaluation on $A$.

**Proof.** We provide a brief sketch. For complete details, see [F-vdP]. For (1), let $f \in \mathcal{O}_P$, and choose $V \in P$ such that $f \in A_V$. If $f \notin \mathcal{M}_P$, then $\|f\|_P > 0$. Thus, there exists $\epsilon > 0$ in $\sqrt{|k^2|}$ such that $V_\epsilon := \{ x \in V : |f(x)| \geq \epsilon \} \in P$. It follows that $f$ is invertible in $V_\epsilon$, and hence $f$ is invertible in $\mathcal{O}_P$. To prove (2), consider $\overline{f} \in k_P$ with $\overline{f} \notin k_P^0$. Choose $V \in P$ such that $f \in A_V$. Then $\{ x \in V : |f(x)| \leq 1 \} \notin P$ since $\overline{f} \notin k_P^0$. If we write $V = \{ x \in V : |f(x)| \leq 1 \} \cup \{ x \in V : |f(x)| \geq 1 \}$, then we conclude that $\{ x \in V : |f(x)| \geq 1 \} \in V$ since $P$ is a prime filter. It follows that $\overline{f}^{-1} \notin k_P^0$.

Finally, for (3) note that if $\psi : F \to K$ is an extension of fields, and $R$ is a valuation ring of $K$, then $\psi^{-1}(R)$ is a valuation ring of $F$. Therefore $B$ is a valuation ring of $\text{Frac}(A/p)$. The image of $A^0$ in $\mathcal{O}_P$ is contained in $\mathcal{O}_P^0$, and thus in $B$. Finally, if $f \in \mathcal{O}_P^0$ is contained in $\cap_{n=1}^\infty \pi^n \mathcal{O}_P^0$, then $\pi^{-n} f \in \mathcal{O}_P^0$ for all $n$ and thus $\|\pi^{-n} f\|_P$ is bounded. So $\|f\|_P = 0$, showing that $\cap_{n=1}^\infty \pi^n \mathcal{O}_P$ is contained in $\mathcal{M}_P$. Thus $\cap_{n=1}^\infty \pi^n k_P^0 = 0$, implying $\cap_{n=1}^\infty \pi^n B = 0$. \hfill \Box

Given a prime filter $P \in \mathcal{P}(X)$, we can construct a continuous semivaluation, as in (3) of the lemma. Next, given a continuous semivaluation $(p, B)$, we can associate to it the prime filter $P = \left\{ \begin{array}{l} \text{special } V \subseteq X : V \ni R(f_0; f_1, \ldots, f_r) \text{ with } \overline{f}_i/\overline{f}_0 \in B \end{array} \right\}$ for some choice of $f_0, \ldots, f_r$ generating the unit ideal.

**Theorem 1.**

1. The above association gives a bijection between $\mathcal{P}(X)$ and $\text{Val}(X)$.

2. Let $\mathcal{M}(X)$ denote the maximal filters on $X$. Under this bijection, $\mathcal{M}(X)$ corresponds to the set of continuous height 1 semivaluations on $A$.

3. For any continuous semivaluation $(p, B)$ on $A$, $\text{ht}(B) \leq \dim(A) + 1$.

4. The retraction map $\mathcal{P}(X) \to \mathcal{M}(X)$ is given by $(p, B) \mapsto (p, B')$, where $B' = B_q$ and $q = \sqrt[\pi]{B}$.

**Proof.** See [F-vdP]. \hfill \Box
We conclude with a concrete example of type V points on the unit disk $\mathbb{B}_k^1$. For any $r \in \sqrt{|k^\times|}$ with $0 < r \leq 1$, define $\Gamma := \mathbb{R}_{>0} \times \mathbb{Z}$ with the unique total order satisfying $r' < \gamma_r < r$ for any $r' < r$. For any $a \in k$, set $D^{-} := D(a, r)^{-}$. Then we can define a continuous height 2 valuation on $k(T)$ by

$$\left. \sum_{n=0}^{\infty} a_n(T - a)^n \right|_{D^-} := \max_n |a_n|_{\gamma_r}.$$ 

Let $B := \{ f \in \text{Frac}(k(T)) : |f|_{D^-} \leq 1 \}$. Then it is easy to check that $(0, B)$ defines a continuous semivaluation on $k(T)$, giving rise to a type V point on $\mathbb{B}_k^1$.

References

Math 274: Non-Archimedean Geometry

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The Reduction Map

Let \( A \) be a commutative, non-Archimedean Banach algebra. Recall that we have the spectral seminorm \( \rho \) on \( A \), defined by

\[
\rho(f) = \lim_{n \to \infty} \sqrt[n]{||f^n||} = \max_{x \in M(A)} |f(x)|
\]

for any \( f \in A \). We also write \( \rho(f) \) as \( ||f||_{\text{sup}} \).

Recall that \( A^\circ \) denotes the subring of \( A \) consisting of its power-bounded elements:

\[
A^\circ := \{ f \in A : ||f||_{\text{sup}} \leq 1 \} = \{ f \in A : \exists M \text{ s.t. } ||f^n|| \leq M \forall n > 0 \}.
\]

\( A^\circ \) contains an ideal \( A^{\circ \circ} \) consisting of the topologically nilpotent elements of \( A \):

\[
A^{\circ \circ} := \{ f \in A : ||f||_{\text{sup}} < 1 \} = \{ f \in A : ||f^n|| \to 0 \text{ as } n \to \infty \}.
\]

Note that \( A^{\circ \circ} \) is not necessarily a maximal ideal in \( A^\circ \). We define the reduction of \( A \) to be the quotient ring

\[
\tilde{A} := A^\circ / A^{\circ \circ}.
\]

The construction of \( \tilde{A} \) is functorial: if \( \phi : A \to B \) is a bounded homomorphism of commutative non-Archimedean Banach algebras, then \( \phi \) sends \( A^\circ \) to \( B^\circ \) and \( A^{\circ \circ} \) to \( B^{\circ \circ} \), so we get an induced ring homomorphism \( \bar{\phi} : \tilde{A} \to \tilde{B} \).

We will now construct a canonical map

\[
\mathcal{M}(A) \to \text{Spec}(\tilde{A}).
\]

Given an element \( x \in \mathcal{M}(A) \) with completed residue field \( \mathcal{H}(x) \), we have a corresponding character \( \chi_x : A \to \mathcal{H}(x) \). With respect to the norm on \( \mathcal{H}(x) \)
inherited from $\mathcal{A}$, $\chi_x$ is a bounded homomorphism of Banach rings. So as above, we get an induced homomorphism

$$\tilde{\chi}_x : \tilde{A} \rightarrow \tilde{\mathcal{H}(x)},$$

where $\tilde{\mathcal{H}(x)}$ denotes the residue field of $\mathcal{H}(x)$. In particular, the kernel of $\tilde{\chi}_x$ is a prime ideal in $\tilde{A}$, so this defines a canonical reduction map

$$\pi : \mathcal{M}(\mathcal{A}) \rightarrow \text{Spec}(\tilde{\mathcal{A}})$$

$$x \mapsto \ker(\tilde{\chi}_x).$$

**Example 1.** Let $\mathcal{A} = k\langle T \rangle$ ($k$ complete, non-Archimedean, algebraically closed), so $\mathcal{M}(\mathcal{A})$ is the Berkovich analytification of the unit ball. More precisely,

$$\mathcal{A} = \left\{ \sum_{n=0}^{\infty} a_n T^n : a_n \in k, |a_n| \rightarrow 0 \text{ as } n \rightarrow \infty \right\}.$$  

The norm on $\mathcal{A}$ is the sup norm: $||\sum a_n T^n|| = \max_{n \geq 0} |a_n|$. In this case, $\mathcal{A}^o = \{ \sum a_n T^n : a_n \in k^o, |a_n| \rightarrow 0 \}$. Thus for $\sum a_n T^n \in \mathcal{A}^o$, we have $a_n \in k^{\circ \circ}$ for $n \gg 0$, so we conclude that $\tilde{\mathcal{A}}$ is the polynomial ring $\tilde{k}[T]$, where $\tilde{k}$ is the residue field of $k$.

Thus $\text{Spec}(\tilde{\mathcal{A}})$ is the affine line over $\tilde{k}$, and we have the reduction map $\pi : \mathcal{M}(\mathcal{A}) \rightarrow \mathbb{A}^1_k$. Working through the definitions, we can see that this map sends the Gauss point $\zeta_{\text{Gauss}}$ to the generic point of $\mathbb{A}^1_k$. If $x \neq \zeta_{\text{Gauss}}$, then there exists some $a \in k^o$ such that $|T - a|_x < 1$, and it turns out that $\pi(x) = \tilde{a} \in \tilde{k}$, viewing $\tilde{k}$ as the set of closed points in $\mathbb{A}^1_k$.

Now we recall some facts about affinoid algebras. These may be found in [BGR]. Throughout, let $k$ be a complete, non-Archimedean field (possibly trivially valued).

Recall that a **strictly $k$-affinoid algebra** is a quotient of some Tate algebra $T_n = k\langle T_1, ..., T_n \rangle$ by some ideal (which is automatically closed), equipped with the residue norm induced by the sup norm on $T_n$. The residue norm is not canonical, but any two presentations of an algebra as a quotient of a Tate algebra yield equivalent residue norms. For the rest of today’s lecture, $\mathcal{A}$ will denote a strictly $k$-affinoid algebra.

**Proposition 1.** (1) $\tilde{\mathcal{A}}$ is a reduced, finitely generated $\tilde{k}$-algebra.

(2) If $\mathcal{A}$ is reduced, then the spectral seminorm $\rho$ is a norm, and is equivalent to the given Banach norm $||\cdot||$ on $\mathcal{A}$.

(3) $\mathcal{A}$ is Noetherian.

(4) (“Maximum modulus principle for strictly $k$-affinoid algebras”) For any $f \in \mathcal{A}$, $\rho(f) \in |k^{\text{alg}}| = \sqrt{|k|}$. 

2
Proof. For (1), cf. [BGR] §1.2, Prop. 7 and §6.3, Cor. 3. For (2) and (4), cf. [BGR] §6.2, Prop. 4. For (3), cf. [BGR] §6.1, Prop. 3.

Proposition 2. (1) If $f_1, \ldots, f_r \in \mathcal{A}$ and $I = \langle f_1, \ldots, f_r \rangle \subset \tilde{\mathcal{A}}$, then for the reduction map $\pi : \mathcal{M}(\mathcal{A}) \to \text{Spec}(\tilde{\mathcal{A}})$ we have

$$\pi^{-1}(V(I)) = \{x \in \mathcal{M}(\mathcal{A}) : |f_i(x)| < 1 \forall i \}.$$

(2) If $f \in \mathcal{A} \setminus \mathcal{A}^\infty$, then

$$\pi^{-1}(D(f)) = \{x \in \mathcal{M}(\mathcal{A}) : |f(x)| = 1 \}.$$

(3) If $f \in \mathcal{A} \setminus \mathcal{A}^\infty$, then $\pi^{-1}(D(f))$ is a nonempty, compact subset of $\mathcal{M}(\mathcal{A})$.

Proof. The first two statements are left as easy exercises. To see that $\pi^{-1}(D(f))$ is nonempty for $f \in \mathcal{A} \setminus \mathcal{A}^\infty$, note that for such an $f$ we have $|f|_{\text{sup}} = 1$, so by the maximum modulus principle there is some $x \in \mathcal{M}(\mathcal{A})$ with $|f(x)| = 1$. 

Corollary 1. The reduction map $\pi : \mathcal{M}(\mathcal{A}) \to \text{Spec}(\tilde{\mathcal{A}})$ is anti-continuous: the inverse image of an open set is closed.

Proof. $\tilde{\mathcal{A}}$ is Noetherian, so any open subset of $\text{Spec}(\tilde{\mathcal{A}})$ is a finite union of sets of the form $D(f)$. In particular the inverse image of any open set is compact by the previous lemma, hence closed.

Lemma 1. If $p \in \text{Spec}(\tilde{\mathcal{A}})$ is a minimal prime ideal, then $\pi^{-1}(p)$ is closed and nonempty.

Proof. Since $p$ is minimal, we have

$$\bigcap_{f \notin p} D(f) = \{p\},$$

hence

$$\pi^{-1}(p) = \bigcap_{f \notin p} \pi^{-1}(D(f)).$$

This is an intersection of closed sets, hence closed.

For any finite subset $I \subset \tilde{\mathcal{A}} \setminus p$,

$$\bigcap_{f \in I} \pi^{-1}(D(f)) = \pi^{-1}(D(\prod_{f \in I} f)),$$

which is closed and nonempty. Since $\mathcal{M}(\mathcal{A})$ is compact, it follows that the entire intersection

$$\bigcap_{f \notin p} D(f)$$

is nonempty. 

3
Next time we will see that \( \pi : \mathcal{M}(\mathcal{A}) \to \text{Spec}(\mathcal{A}) \) is surjective, and that the inverse image of a minimal prime \( \mathfrak{p} \) is a single point. The preimage \( \Gamma(\mathcal{A}) \subset \mathcal{M}(\mathcal{A}) \) of all the minimal prime ideals in \( \text{Spec}(\mathcal{A}) \) is called the **Shilov boundary of** \( \mathcal{M}(\mathcal{A}) \).
1 Reminders from Last Time

Let $k$ be as usual, and let $A$ be a strictly $k$-affinoid algebra. We define $\bar{A} := A^0/A^{00}$; this is a finitely generated $k$ algebra. For $x \in \mathcal{M}(A)$, we have a map

$$\chi_x : A \to \mathcal{H}(x),$$

inducing a map

$$\bar{\chi}_x : \bar{A} \to \overline{\mathcal{H}(x)}.$$ 

This allows us to define the map

$$\pi : \mathcal{M}(A) \to \text{Spec} (\bar{A})$$

by

$$\pi(x) = \ker(\bar{\chi}_x) \in \text{Spec} A.$$

**Example 1.** If $A = k \langle T \rangle$, then

$$\pi : (B_k)^{\text{an}} \to A_k^1$$

sends the Gauss point to the generic point, and sends $x \in (D(a,1)^{\text{an}})$ to $\overline{a}$, a closed point.

2 Properties of $\pi$

**Theorem 1** (Berkovich). Let $A$ be a strictly $k$-affinoid algebra.

1. $\pi : \mathcal{M}(A) \to \text{Spec} (\bar{A})$ is surjective.

2. For each generic point $\bar{x} \in \text{Spec} (\bar{A})$ (that is, each point corresponding to a minimal prime), $\pi^{-1}(\bar{x})$ is a singleton $\{x\}$. We call $x$ the Shilov point corresponding to $\bar{x}$.

To prove (1), we will use the following result from [BGR].

**Proposition 1.** The map

$$\text{Max}(A) \to \text{Max} (\bar{A})$$

is surjective. (This map is defined since if $x \in \text{Max}(A)$, then $\mathcal{H}(x)$ is a finite extension of $k$; hence $\ker(\bar{\chi}_x)$ is a maximal ideal.)
Proof of Part (1) of the Theorem. Note that if \( K/k \) is any valued field extension, then we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}(A_K) & \longrightarrow & \text{Spec} \left( \tilde{A}_K \right) \\
\downarrow & & \downarrow \\
\mathcal{M}(A) & \longrightarrow & \text{Spec} \left( \tilde{A} \right).
\end{array}
\]

We claim that for any point \( x \in \tilde{X} := \text{Spec}(\tilde{A}) \), there exists a complete non-Archimedean field extension \( K/K \) and a closed point \( y \) of \( \text{Spec} \left( \tilde{A}_K \right) \) lying over \( x \). Once we’ve shown this, the claim follows from the surjectivity of \( \text{Max}(A) \to \text{Max} \left( \tilde{A} \right) \).

In order to prove the claim, it suffices to prove the following two statements:

(a) Given \( x \in \tilde{X} \), there exists \( K/k \) such that the natural map \( \text{Spec}(\tilde{A} \otimes_k \tilde{K}) \to \text{Spec}(\tilde{A}) \) has a closed point in the fiber over \( x \).

(b) For every \( K/k \), the natural map \( \text{Spec}(\tilde{A}_K) \to \text{Spec}(\tilde{A} \otimes_k \tilde{K}) \) is surjective.

Statement (a) can be proved by induction as follows. It is easy to find an extension \( F/\tilde{k} \) such that

\[
\text{Spec}(\tilde{A} \otimes_k F) \to \text{Spec}(\tilde{A})
\]

has a closed point in the fiber over \( x \); in particular, take \( F = \tilde{k}(x) \) to be the residue field of \( x \). Since \( F \) is finitely generated over \( k \), we may invoke the following general fact: If \( F/\tilde{k} \) is any finitely generated field extension, then there exists a complete valued field extension \( K/k \) such that \( \tilde{K} \cong F \) as extensions of \( \tilde{k} \).

(Proof: It suffices to treat separately the cases (i) \( F/\tilde{k} \) finite and (ii) \( F = \tilde{k}(T)/\tilde{k} \) purely transcendental of transcendence degree 1. Case (i) is an easy exercise, and case (ii) follows by letting \( K \) be the completed residue field of the Gauss point of \( M(\kappa(T)) \).)

To establish (b), we claim that the natural map \( \tilde{A} \otimes_{\tilde{k}} \tilde{K} \to \tilde{A}_K \) is finite and injective (and therefore \( \text{Spec}(\tilde{A}_K) \to \text{Spec}(\tilde{A} \otimes_k \tilde{K}) \) is surjective by the going-up theorem). Since \( \tilde{A}_K \) is of finite type over \( \tilde{K} \) and a homomorphism of \( K \)-algebras is finite if and only if it is integral and finite type, it suffices to prove that the natural injective map \( B := \tilde{A} \otimes_{\tilde{k}} \tilde{K} \to \tilde{A}_K \) is integral (since tensoring with \( \tilde{K} \) then gives an integral map \( \tilde{A} \otimes_{\tilde{k}} \tilde{K} \to \tilde{A}_K \)). By ([BGR], 6.1.2/1(ii)), there is a finite injective \( K \)-algebra homomorphism \( K \langle X_1, \ldots, X_d \rangle \to A_K \) such that \( K^0 \langle X_1, \ldots, X_d \rangle \) is contained in \( B \), and by ([BGR], 6.3.4/1) the induced map \( K^0 \langle X_1, \ldots, X_d \rangle \to A_K \) is integral. It follows that \( A_K^0 \) is integral over \( B \). This completes the proof.

\[ \square \]

Proof of Part (2) of the Theorem. Let \( \hat{x} \in \tilde{X}_{\text{gen}} \), where \( \tilde{X} = \text{Spec}(\tilde{A}) \). Last time we proved that \( \pi^{-1}(\hat{x}) \) is nonempty. (This is also implied by part (1).)

Let \( x \in \pi^{-1}(\hat{x}) \). Without loss of generality, \( A \) is reduced, as this doesn’t change the spectrum topologically.

For a first case, we’ll assume that \( \tilde{A} \) is an integral domain (i.e. there exists a unique generic point of \( \tilde{X} \)). We claim that for all \( f \in A \), \( |f(x)| = ||f||_{\text{sup}} \). (This uniquely determines \( x \).)

We always have \( |f(x)| \leq ||f||_{\text{sup}} \), so without loss of generality we may take \( ||f||_{\text{sup}} > 0 \). Then there exists \( a \in k^* \) and \( n \in \mathbb{N} \) such that \( ||f||_{\text{sup}} = |a| \), since \( ||f||_{\text{sup}} \in \sqrt{|k^*|} \). Let \( g = a^{-1} f^n \), so that \( ||g||_{\text{sup}} = 1 \). Hence
$g \in A^0 \setminus A^{00}$, so $\tilde{g} \neq 0$. As $\tilde{A}$ is an integral domain, $\tilde{g}(\tilde{x}) \neq 0$. So $x \in \pi^{-1}(D(\tilde{g}))$, implying $|g(x)| = 1$, giving us $||g||_{\text{sup}} = |g(x)|$. Since $\tilde{A}$ is a domain, $|| \cdot ||_{\text{sup}}$ is multiplicative, so we may “unscale” this equality to get $||f||_{\text{sup}} = |f(x)|$.

Now for the general case (meaning $\tilde{A}$ might not be an integral domain). Choose $f \in A^0$ such that $\tilde{f} \in \tilde{A}$ vanishes at all points of $\tilde{X}_{\text{gen}}$ except for $\tilde{x}$.

Let $B = A(T)/(1 - Tf)$. Then

- $B$ is a (strict) $k$-affinoid algebra,
- $\tilde{B}$ is $\tilde{A}_f$, an integral domain, and
- $\mathcal{M}(B)$ is canonically homeomorphic to $\pi^{-1}(D(\tilde{f})) \subset M(A)$ and $\pi|_{\mathcal{M}(B)} = \pi|_{\mathcal{M}(B)}$.

Hence we have reduced to the previous case, completing the proof. \hfill \square

**Remark 1.** A reasonable question that arises in the above proof is whether or not $A$ is an integral domain if and only if $\tilde{A}$ is an integral domain. The answer is no. For an example, we may consider the affinoid subdomain $\{1 \leq |z| \leq r\}$, where $r > 1$ is in the value group. In this case we have that $A$ is an integral domain (as an affinoid algebra), but $\tilde{A}$ yields two irreducible components (essentially looking like two lines intersecting at a point). This example will be worked out in more detail in the next set of notes.
1 Surjectivity in the Diagram From Last Time

Let \( k \) be as usual, and let \( A \) be a strictly \( k \)-affinoid algebra. Let \( K/k \) be a valued field extension. Last time we considered the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}(A_K) & \longrightarrow & \text{Spec} \left( \mathcal{A}_K \right) \\
\downarrow & & \downarrow \\
\mathcal{M}(A) & \longrightarrow & \text{Spec} \left( \mathcal{A} \right)
\end{array}
\]

and proved that the horizontal arrow \( \pi : \mathcal{M}(A) \rightarrow \text{Spec} \left( \mathcal{A} \right) \) is surjective.

**Remark 1.** The map \( \mathcal{M}(A_K) \rightarrow \mathcal{M}(A) \) is also surjective.

**Proof.** Let \( L \) be a valued field extension of \( k \), and let \( A \rightarrow L \) be given. Choose \( L' \) extending both \( L \) and \( K \) (for the existence of \( L' \), see the notes from February 8). This gives a map from \( A \) to \( L' \) and from \( K \) to \( L' \), inducing a map \( A_K \rightarrow L' \) since \( A_K = A \otimes_k K \). This gives us the desired surjectivity.

\[
\begin{array}{ccc}
A \otimes_k K = A_K & \longrightarrow & L' \\
\uparrow & & \uparrow \\
A & \longrightarrow & K
\end{array}
\]

2 Adding Adics to the Picture

Let \( A \) be as before. Let \( X = \text{Max}(A) \), and \( \overline{X} = \text{Max}(\overline{A}) \). We have a map \( \text{red} : X \rightarrow \overline{X} \), which is surjective (due to Tate). Now, \( X = \text{Max}(A) \subset \mathcal{M}(A) = M(X) \subset P(X) \), where \( M(X) \) denotes the maximal filters on \( L \), the lattice of special subsets in \( X \), and where \( P(X) \) is the prime filters on \( L \). Now, we have a projection map \( P(X) \rightarrow P(\overline{X}) \) which sends a prime filter \( \mathcal{F} \) on \( X \) to \( \mathcal{F} \), where \( U \in \mathcal{F} \) if and only if \( \text{red}^{-1}(U) \in \mathcal{F} \). By a well-known result (cf. [FvdP]), \( P(\overline{X}) \cong \overline{X} := \text{Spec}(\overline{A}) \). This is summarized in the following diagram.
In general, the set has two points. These correspond to the inner and outer boundaries of the annulus.

We have Example 2. theorem, each function attains its maximum at the Gauss point.

Example 1. in the previous set of notes), this implies the general case.

<table>
<thead>
<tr>
<th>Proof of the first claim.</th>
</tr>
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</table>
| Let \( f \in \mathcal{A} \), then there exists \( x \in X_{\text{gen}} \) such that \( f(x) \neq 0 \). Let \( x = \pi^{-1}(\hat{x}) \). Then \( \|f(x)\| = \|f\|_{\text{sup}} \), establishing the claim in this case. By scaling \( f \) to have norm 1 and then unscaling (as in the previous set of notes), this implies the general case. \( \square \)

However, we come across the curiosity that while \( X^{\text{ad}} \) to \( \text{Spec}(\mathcal{A}) \) is continuous, \( X^{\text{an}} \) to \( X^{\text{an}} \) is anticontinuous. This occurs because \( M(X) \) has the quotient topology (induced by the surjection \( P(X) \to M(X) \)), rather than the subspace topology. Also interesting is the fact that \( X^{\text{an}} \) (not just \( X^{\text{ad}} \)) surjects onto \( \text{Spec}(\mathcal{A}) \).

### 3 Shilov Boundaries

Let \( \pi : \mathcal{M}(\mathcal{A}) \to \text{Spec}(\mathcal{A}) \) be the restriction of \( \pi \) to \( M(X) = \mathcal{M}(\mathcal{A}) \), and let \( \Gamma := \pi^{-1}(\{\text{generic points}\}) \).

From last time, each generic point has one point in its preimage, so \( \Gamma \) is a finite set. We call \( \Gamma \) the **Shilov boundary** of \( \mathcal{M}(\mathcal{A}) \).

**Theorem 1.** For every \( f \in \mathcal{A} \),

\[
\sup_{x \in \mathcal{M}(\mathcal{A})} |f(x)| = \max_{x \in \Gamma} |f(x)|.
\]

Moreover, \( \Gamma \) is the unique minimal such subset of \( \mathcal{M}(\mathcal{A}) \) with this property (meaning that \( \Gamma \) is the Shilov boundary).

**Proof of the first claim.** If \( \|f\|_{\text{sup}} = 1 \), then there exists \( \hat{x} \in X_{\text{gen}} \) such that \( \hat{f}(\hat{x}) \neq 0 \). Let \( x = \pi^{-1}(\hat{x}) \). Then \( |f(x)| = \|f\|_{\text{sup}} \), establishing the claim in this case. By scaling \( f \) to have norm 1 and then unscaling (as in the previous set of notes), this implies the general case. \( \square \)

**Example 1.** Let \( X = (\mathbb{B}^1)^n = \mathcal{M}(k\langle T \rangle) \), so that \( \hat{X} = \text{Spec}(\mathbb{A}^1_k) \) and \( \Gamma = \{\text{the Gauss point}\} \). By the theorem, each function attains its maximum at the Gauss point.

**Example 2.** Let \( X \) be the closed annulus \( r \leq |x| \leq 1 \), where \( r \in \sqrt{|k^x|} \) and \( 0 < r < 1 \). Explicitly, \( X \) is

\[
\{x \in \mathcal{M}(k\langle T \rangle) \mid |T|_x \leq 1, |T|_x \geq r \} = \mathcal{M}(k\langle T, S \rangle/(TS - r)).
\]

We have \( \mathcal{A} = k\langle T, S \rangle/(TS - r) \), and \( \tilde{\mathcal{A}} = \hat{k}[T, S]/(TS) \). \( \tilde{\mathcal{A}} \) has two irreducible components, giving us that \( \Gamma \) has two points. These correspond to the inner and outer boundaries of the annulus.

In general, the set

\[
\{x \in \mathcal{M}(\mathcal{A}) \mid |f_i(x)| \leq r_i, |g_j(x)| \geq s_j \}_{i,j}
\]
(with $i$ and $j$ taking on finitely many values, and the $r_i$'s and $s_j$'s are in the value group) corresponds to

$$\mathcal{M} \left( A \langle r^{-1}X, sY \rangle / (X_i - f_i, g_j Y_j - 1) \right),$$

where

$$A \langle r_1^{-1}T_1, \ldots, r_n^{-1}T_n \rangle = \left\{ f = \sum a_{\nu} T^\nu \mid \|a_{\nu} T^\nu\| \to 0 \text{ as } \|\nu\| \to 0 \right\}.$$
Affinoids in the sense of Berkovich

In the classical setting of [BGR], [Tate], etc., one works solely with (strictly) $k$-affinoid algebras of the form $k\langle T_1, \ldots, T_n \rangle / \mathfrak{a}$, or equivalently of the form

$$k\langle r_1^{-1}T_1, \ldots, r_n^{-1}T_n \rangle / \mathfrak{a}$$

with $r_1, \ldots, r_n \in \sqrt{|k^*|}$, where $k$ is a complete non-Archimedean field which is non-trivially valued. (Recall that $k\langle r_1^{-1}T_1, \ldots, r_n^{-1}T_n \rangle$ denotes the algebra of power series $\sum a_IT^I$ with $|a_I|r^I \to 0$ as $|I| \to \infty$.)

Berkovich generalizes this in two ways. First, one allows the possibility that $k$ is trivially valued; second, one allows quotients as above for arbitrary $r_1, \ldots, r_n \in \mathbb{R}_{>0}$. For the rest of today, $k$ will denote an arbitrary complete non-Archimedean field.

**Definition 1.** A Banach $k$-algebra $\mathcal{A}$ is *affinoid* if there exists an admissible surjection $k\langle r_1^{-1}T_1, \ldots, r_n^{-1}T_n \rangle \twoheadrightarrow \mathcal{A}$ for some $r_1, \ldots, r_n \in \mathbb{R}_{>0}$. $\mathcal{A}$ is *strictly $k$-affinoid* if we can choose $r_1, \ldots, r_n \in \sqrt{|k^*|}$.

Here a surjective homomorphism from a Banach $k$-algebra onto $\mathcal{A}$ is considered *admissible* if the Banach norm on $\mathcal{A}$ is equivalent to the residue norm coming from the given surjection.

**Example 1.** Let $r \in \mathbb{R}_{>0}$. We define

$$K_r := \left\{ \sum_{i=-\infty}^{\infty} a_iT^i : a_i \in k, |a_i|r^i \to 0 \text{ as } |i| \to \infty \right\}.$$ 

$K_r$ is a (commutative) Banach $k$-algebra with a multiplicative norm $\| \cdot \|$ defined by

$$\left\| \sum_{i \in \mathbb{Z}} a_iT^i \right\| := \max_{i \in \mathbb{Z}} \{|a_i|r^i\}.$$
Fact: $K_r$ is a (usually non-strict) $k$-affinoid algebra, and if $r \notin \sqrt{|k^*|}$ then $K_r$ is a field.

In fact, let $x$ be the type III point of $k^1_{\text{Berk},k}$ corresponding to the closed disk $D(0,r)$.

Exercise: $K_r \cong \mathcal{H}(x)$, the completed residue field of $x$.

To verify that $K_r$ is $k$-affinoid, one defines a homomorphism

$$k(r^{-1}T_1, rT_2)/(T_1T_2 - 1) \to K_r$$

by sending $T_1 \mapsto T$ and $T_2 \mapsto T^{-1}$, and then one checks that this is an isomorphism. (Note that if instead we had $r \in \sqrt{|k^*|}$, then the algebra defined on the left-hand side above would correspond to the circle of radius $r$ centered at the origin. For $r \notin \sqrt{|k^*|}$ the rigid space corresponding to this algebra has no points in the classical sense.)

Why is $K_r$ a field? One can show that the norm $|| \cdot ||$ defined above is the unique bounded multiplicative norm on $K_r$, and then use this fact to deduce that $K_r \cong \mathcal{H}(x)$.

More generally, given $r_1,\ldots,r_n \in \mathbb{R}_{>0}$ whose images in $\mathbb{R}_{>0}/\sqrt{|k^*|}$ are linearly independent over $\mathbb{Q}$, we have a $k$-affinoid algebra

$$K_{r_1,\ldots,r_n} := K_{r_1} \otimes_k \cdots \otimes_k K_{r_n}.$$

This is a complete non-Archimedean field with value group

$$|K_{r_1,\ldots,r_n}^*| = \langle |k^*|, r_1,\ldots,r_n \rangle.$$

This immediately yields:

**Lemma 1.** For every $k$-affinoid algebra $\mathcal{A}$, there exists a (non-trivially valued) complete non-Archimedean field extension $\bar{K} = K_{r_1,\ldots,r_n}$ of $k$ such that $\mathcal{A} \otimes_k \bar{K}$ is strictly $K$-affinoid.

**Proof.** Choose an admissible surjection $k(r_1^{-1}T_1,\ldots,r_n^{-1}T_n) \to \mathcal{A}$ and let $K = K_{r_1,\ldots,r_n}$.

**Proposition 1.** Let $\mathcal{A}$ be a $k$-affinoid algebra. Then $\mathcal{A}$ is Noetherian and every ideal $a \subseteq \mathcal{A}$ is closed.

**Proof.** By the above lemma and Tate’s corresponding theorems for strict affinoids, it suffices to show that if $r \notin \sqrt{|k^*|}$ and $\mathcal{A} \otimes_k K_r$ is Noetherian with all of its ideals closed, then the same holds for $\mathcal{A}$.

Identify $\mathcal{A}$ with its image in $\mathcal{A} \otimes_k K_r$, and let $a \subseteq \mathcal{A}$ be an ideal. Then the ideal $a(\mathcal{A} \otimes_k K_r)$ of $\mathcal{A} \otimes_k K_r$ is finitely generated, say by elements $f_1,\ldots,f_m$. Without loss of generality we may assume each $f_i$ lies in $a$. 

2
Let $f \in a$. Then there exist $g_1, \ldots, g_m \in A \otimes_k K_r$ such that $f = \sum_{i=1}^{m} f_i g_i$. Write

$$g_i = \sum_{j=-\infty}^{\infty} g_{ij} T^j$$

with each $g_{ij} \in A$ and $||g_{ij}|| r^j \to 0$ as $|j| \to \infty$. Since $||T|| = r \notin \sqrt{|k^*|}$, we in fact have $f = \sum_{i=1}^{m} g_{i0} f_i$. Therefore $a$ is generated by $f_1, \ldots, f_m$, so we conclude that $A$ is Noetherian.

This also shows that $a = A \cap a(A \otimes_k K_r)$. Both $A$ and $a(A \otimes_k K_r)$ are closed in $A \otimes_k K_r$, so $a$ is closed in $A$.

Next time, we will define $k$-affinoid subdomains of $M(A)$ for arbitrary $k$-affinoid algebras $A$, and we will see that an analogue of Tate’s acyclicity theorem holds in this more general setting: given a finite cover of $M(A)$ by affinoid subdomains, the corresponding Čech complex is exact and admissible.
1 Affinoid domains

Definition 1. Let \( k \) be a complete non-Archimedean field (possibly trivially valued), \( A \) a affinoid \( k \)-algebra, \( X = \mathcal{M}(A) \). A subset \( U \subset \mathcal{M}(A) \) is a \( k \)-affinoid subdomain if there exists a bounded homomorphism \( A \to A_U \) of \( k \)-affinoid algebras with
\[
\mathcal{M}(A_U) \to U \subset \mathcal{M}(A)
\]
such that whenever \( A \to B \) is a bounded homomorphism of \( k \)-affinoid algebras with
\[
\mathcal{M}(B) \to U \subset \mathcal{M}(A),
\]
there exists the unique bounded homomorphism \( A_U \to B \) such that the diagram
\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A_U & \longrightarrow &
\end{array}
\]
commutes.

Remark 1. Any \( k \)-algebra homomorphism \( A \to B \) between strictly \( k \)-affinoid algebras is bounded ([BGR], 6.1.3/1). This is not necessarily true in the non-strict case (cf. [Berk] Remark. 2.1.13).

Remark 2. The definition implies that \( A_U \) and \( A \to A_U \) are unique up to isomorphism.

Remark 3. The definition does not immediately imply that \( M(A_U) \to U \), but this is true. The idea of the proof is to reduce to the strict case, which is proved in [BGR]. In order to do this reduction, we need to use the surjectivity of \( M(A_U \otimes_k K) \to M(A_U) \) (cf. [Berk], Prop. 2.2.4).

Exercise 1. Weierstrass (resp. Laurent, rational) subdomains are \( k \)-affinoid subdomains.

Definition 2. A strict \( k \)-affinoid subdomain of a strictly \( k \)-affinoid space is defined similarly but with \( A \to B \) (bounded) homomorphism of strictly \( k \)-affinoid algebras and \( A_U \) strict.

Remark 4. It is not clear a priori that a strictly \( k \)-affinoid subdomain is a \( k \)-affinoid subdomain. But, it is true! To prove this, one needs the Gerritzen-Grauert theorem, which says that every (strictly) \( k \)-affinoid subdomain of a (strict) \( k \)-affinoid space is a finite union of (strict) rational subdomains. (It is an exercise that rational subdomains have a stronger universal property). For the moment, we assume the Gerritzen-Grauert theorem.

Remark 5. Another consequence of the Gerritzen-Grauert theorem is that if \( U \subset \mathcal{M}(A) \) is a (strictly) \( k \)-affinoid subdomain and \( x \in U = M(A_U) \subset M(A) \), then \( \mathcal{H}_{\mathcal{M}(A)}(x) \) and \( \mathcal{H}_{\mathcal{M}(A_U)}(x) \) coincide. Temkin found an independent proof of this fact not using the Gerritzen-Grauert theorem, and he uses this to give a new proof of the Gerritzen-Grauert theorem.
2 Tate’s acyclicity theorem

**Proposition 1.** Let $X$ be a Banach space over $k$, $K = K_r$ as in the last lecture where $r \notin \sqrt{|K^\times|}$.

1. We have an isomorphism of $K$-Banach spaces

   $$X \hat{\otimes}_k K \cong \{ f = \sum_{i=-\infty}^{\infty} x_i T^i \mid x_i \to X, \| x_i \| r^i \to 0 \text{ as } i \to \pm \infty \}$$

   where the norm of the second algebra is defined by $\| f \| = \sup_i \| x_i \| r^i$.

2. The canonical map $X \to X \hat{\otimes}_k K$ is an isometric embedding.

3. A sequence of bounded homomorphisms $X \overset{f}{\to} Y \overset{g}{\to} Z$ of $k$-Banach spaces is *exact and admissible*, i.e., $\ker (g) = \operatorname{im}(f)$, and the norm $\| \cdot \|_Y$ on $\ker (g)$ is equivalent to the quotient seminorm on $\operatorname{im}(f)$ induced by $\| \cdot \|_X$, if and only if

   $$X \hat{\otimes}_k K \to Y \hat{\otimes}_k K \to Z \hat{\otimes}_k K$$

   is exact and admissible.

**Next time:** Given a finite covering of a $k$-affinoid space $X = M(A)$ by $k$-affinoid subdomains $V_i$, the corresponding Čech complex

$$0 \to A \to \prod_i A_{V_i} \to \prod_{ij} A_{V_i \cap V_j} \to \cdots$$

is exact and admissible.
Math 274: Non-Archimedean Geometry

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The structure sheaf of a Berkovich space

Recall that in algebraic geometry, to define the structure sheaf $O_X$ of an affine scheme $X = \text{Spec}(R)$, we set

$$O_X(D(f)) := R_f$$

for any $f \in R$, where $D(f) = \{ p \in X : f \not\equiv p \}$. The open sets $D(f)$ form a basis for the Zariski topology on $X$, and one proves that this definition extends uniquely to give a sheaf $O_X$ on $X$.

In rigid and Berkovich geometry, this is more complicated. Let $A$ be a $k$-affinoid algebra. Given $f \in A$, the localization $A_f$ is not generally an affinoid algebra. We can replace it with the $k$-affinoid algebra

$$\tilde{A}_f := A(T)/(Tf - 1),$$

so $M(\tilde{A}_f)$ is the Laurent domain $\{ x \in M(A) : |f(x)| \geq 1 \}$. But it is insufficient to only consider domains of this form, because the canonical topology is too fine: the Laurent domains $M(\tilde{A}_f)$ for $f \in A$ do not give a basis for the canonical topology.

Instead, we need to consider more general Laurent domains of the form

$$V = \{ x \in M(A) : |f_i(x)| \leq s_i \text{ for } i = 1, \ldots, m, \ |g_j(x)| \geq t_j \text{ for } j = 1, \ldots, n \}$$

for $f_1, \ldots, f_m, g_1, \ldots, g_n \in A$ and $a_i, b_j \in A$ with $|a_i| = s_i, |b_j| = t_j$. For such a Laurent domain $V$ we set

$$A_V := A(S_1, \ldots, S_m, T_1, \ldots, T_n)/(f_i - a_iS_i, g_jT_j - b_j).$$

It is not immediately clear that $A_V$ is independent of the choice of presentation of $V$. But one checks readily that the natural map $A \to A_V$ has the universal property in the definition of an affinoid domain, so indeed $A_V$ is canonically determined by $V$ and the assignment $V \mapsto A_V$ is functorial in the appropriate sense.

**Remark 1.** It will be important to work with rational domains, rather than only Laurent domains, for two reasons:

1. Being a Laurent subdomain is not a transitive concept: cf. [BGR, §7.2.4] for an example where $W$ is a Laurent subdomain of $V$ and $V$ is a Laurent subdomain of $X$, but $W$ is not a Laurent subdomain of $X$. In contrast, being a rational subdomain is a transitive concept.

2. The Gerritzen-Grauert theorem tells us that every strict $k$-affinoid domain is a finite union of strict rational domains. This statement is no longer true if we replace “rational” with “Laurent.”
**Definition 1.** Let $\mathcal{A}$ be a $k$-affinoid algebra and $X = \mathcal{M}(\mathcal{A})$. Let $U \subseteq X$ be open. We define

$$O_X(U) := \lim_{\longleftarrow} A_V,$$

where the inverse limit is taken over all special sets $V \subseteq U$.

One checks easily that $O_X$ is a presheaf on $X$. In fact it is a sheaf, which is proved using the following two facts:

1. Tate’s acyclicity theorem (which implies $O_X$ is a “$G$-sheaf”).

2. **Exercise:** If $\{U_i\}_{i \in I}$ is an open cover of some open subset $U \subseteq \mathcal{M}(\mathcal{A})$, and $V \subseteq U$ is a special set, then there is a finite open covering $\{V_1, ..., V_m\}$ of $V$ with each $V_i$ contained in some $U_j$, for $j_i \in I$.

A straightforward formal argument, left as an exercise, shows that 1 and 2 imply $O_X$ is a sheaf.

**Remark 2.** Last time we gave two examples of non-admissible covers. One was the cover

$$\mathbb{B}_k^1 = \text{Sp}(\mathcal{A}) = \{z \in k : |z| \leq 1\} = \coprod_{i} D(a_i, 1)^-$$

for $\mathcal{A} = k(T)$, where the disjoint union is over a collection $\{a_i\}$ of representatives of the different residue classes. Here $D(a_i, 1)^- = \{z \in k : |z-a_i| < 1\}$. We may extend this to the subset $\{x \in \mathcal{M}(\mathcal{A}) : |T-a_i|^x < 1\}$ of $\mathbb{A}^1_{\text{Berk}, k}$, which is an open subset in the Berkovich topology. But this is no longer an open cover:

$$(\mathbb{B}_k^1)^{\text{an}} = \mathcal{M}(\mathcal{A}) = \{x \in \mathcal{M}(\mathcal{A}) : |T|^x \leq 1\} \neq \bigcup \{x \in \mathcal{M}(\mathcal{A}) : |T-a_i|^x < 1\},$$

because none of the open subsets on the right contain the Gauss point.
Remark 1. A \textit{k}-affinoid space is defined to be a locally ringed space \((X, \mathcal{O}_X)\) where \(X = \mathcal{M}(A)\), \(A\) is a \textit{k}-affinoid algebra, and \(\mathcal{O}_X\) is the structure sheaf defined in the previous lecture. A morphism \((X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) of \textit{k}-affinoid spaces is defined to be a morphism of the locally ringed spaces which is induced by a bounded homomorphism \(A \to B\) of the corresponding \textit{k}-affinoid algebras. Thus the category of \textit{k}-affinoid spaces is antiequivalent to the category of \textit{k}-affinoid algebras. One can define the category of \textit{k}-analytic spaces analogously. Finally, one can define a (global) \textit{k}-analytic space to be a locally ringed space which is locally isomorphic to a \textit{k}-affinoid space. This is the approach taken in Berkovich’s book, but it isn’t the “modern” definition, which we will discuss in a few weeks.

1 Relative Interior and Relative Boundary

\textbf{Definition 1.} Let \(\phi : Y \to X\) be a morphism of \textit{k}-affinoid spaces \(Y = \mathcal{M}(B)\) and \(X = \mathcal{M}(A)\). The \textit{relative interior} \(\text{Int}(Y/X)\) is the set of all \(y \in Y\) for which there is an admissible surjection \(A(r_1^{-1}T_1, \ldots, r_n^{-1}T_n) \to B\) sending \(T_i \mapsto f_i \in B\), such that \(|f_i(y)| < r_i\) for all \(i\).

The \textit{relative boundary} \(\partial(Y/X)\) is defined to be \(Y \setminus \text{Int}(Y/X)\).

In other words, \(y \in Y\) is in the relative interior of \(Y\) over \(X\) if \(Y\) can be admissibly embedded into \(X\) times a polydisc, in such a way that \(y\) is sent to a point in the “naive interior” of the polydisc. (Here, the “naive interior” of a polydisc \(\{\tilde{y} : |y_i| \leq r_i \forall i\}\) is the set \(\{\tilde{y} : |y_i| < r_i \forall i\}\). This isn’t the topological interior in any reasonable sense.) The point is that giving an admissible surjection \(A(r_1^{-1}T_1, \ldots, r_n^{-1}T_n) \to B\) is equivalent to giving a closed admissible embedding \(\mathcal{M}(B) \hookrightarrow \mathcal{M}(A(r_1^{-1}T_1, \ldots, r_n^{-1}T_n))\), and \(\mathcal{M}(A(r_1^{-1}T_1, \ldots, r_n^{-1}T_n))\) is \(\mathcal{M}(A)\) times a polydisc.

\textbf{Exercise 1.} \(\text{Int}(Y/X)\) is an open subset of \(Y\), and so \(\partial(Y/X)\) is a closed subset of \(Y\).

A \textit{boundaryless} morphism \(Y \to X\) is one for which \(\partial(Y/X) = \emptyset\). This notion will be important later - for example the correct definition of a proper morphism of global analytic spaces will be one that is topologically proper, and boundaryless (once we extend the definition of relative interior and relative boundary to general analytic spaces). It will also make an appearance in Temkin’s Berkovich-theoretic proof of the Gerritzen-Grauert Theorem.

We now state a number of important properties and examples of relative interior and relative boundary, without proof:

1. If \(Y\) is an affinoid domain in \(X\), and we consider the inclusion \(Y \hookrightarrow X\), then \(\text{Int}(Y/X)\) is just the topological interior of \(Y\) as a subset of \(X\). This is relatively straightforward to see in the case when \(Y\) is a rational subdomain, and the general case follows by the Gerritzen-Grauert theorem.
We remark that the inclusion \( Y \hookrightarrow X \) of Berkovich spaces shares features of both open and closed immersions. Topologically, it is a map from one compact space to another, so it is a closed immersion. Algebraically, it acts more like a scheme-theoretic open immersion.

2. A morphism \( \phi : Y \to X \) of \( k \)-affinoid spaces is boundaryless if and only if \( \phi : Y \to X \) is finite, meaning that \( A \to B \) makes \( B \) a finite \( A \)-module. This is interesting because being finite is a global condition, while the definition of relative boundary is more local in nature.

3. If \( X = \mathcal{M}(A) \) and \( Y = \mathcal{M}(B) \) are strictly \( k \)-affinoid spaces, then

\[
\text{Int}(Y/X) = \{ y \in Y \mid \tilde{\chi}_y(\tilde{B}) \text{ is finite over } \tilde{\chi}_x(\tilde{A}) \text{ where } x = \phi(y) \}
\]

where \( \phi : Y \to X \) is a morphism of \( k \)-affinoid spaces, \( \chi_y \) is the character corresponding to \( y \), i.e., the map \( \chi_y : B \to \mathcal{H}(y) \), and \( \tilde{\chi}_y : \tilde{b} \to \tilde{\mathcal{H}}(y) \) is the reduction. The reduction map is functorial, giving a commutative diagram

\[
\begin{array}{ccc}
\tilde{A} & \longrightarrow & \tilde{\chi}_x(\tilde{A}) \\
\downarrow & & \downarrow \\
\tilde{B} & \longrightarrow & \tilde{\chi}_y(\tilde{B}) \\
\end{array}
\]

Saying that \( \tilde{\chi}_y(\tilde{B}) \) is finite over \( \tilde{\chi}_x(\tilde{A}) \) means that the dotted arrow is a finite morphism of \( \tilde{k} \)-algebras. This gives a convenient way of determining the relative interior or boundary in the strict affinoid case. Using Definition 1 by itself, it is difficult to prove that a point belongs to the relative boundary, since one must consider all possible embeddings of \( Y \) into \( X \) times a polydisc.

This fact also shows that finiteness of a map \( A \to B \) can somehow be checked pointwise.

\textbf{Example 1.} Let \( r < 1 \), let \( Y = B^1(0, r) \) be the ball of radius \( r \) centered at 0, let \( X = B^1(0, 1) \) be the unit ball, and consider the inclusion \( Y \hookrightarrow X \). Then \( \partial(Y/X) \) is the topological boundary of \( Y \) in \( X \). By viewing the Berkovich spaces as trees, one can see that this topological boundary consists of a single point, the Gauss point of \( Y \), or equivalently, the type II or type III point associated to the disk of radius \( r \) centered at 0.

\textbf{Definition 2.} If \( Y = \mathcal{M}(B) \) is a \( k \)-affinoid space, the \textit{absolute interior} \( \text{Int}(Y) \) and \textit{absolute boundary} \( \partial(Y) \) are defined to be \( \text{Int}(Y/\mathcal{M}(k)) \) and \( \partial(Y/\mathcal{M}(k)) \), where \( Y \to \mathcal{M}(k) \) is the map induced by \( k \to B \).

4. If \( Y \) is strictly \( k \)-affinoid, then \( \text{Int}(Y) = \pi^{-1}(\text{Max}(\tilde{B})) \) where \( \pi : \mathcal{M}(B) \to \text{Spec } \tilde{B} \) is the reduction map and \( \text{Max}(\tilde{B}) \) is the set of closed (maximal) points of \( \text{Spec } \tilde{B} \). By contrast, the Shilov boundary \( \Gamma(Y) \) is the preimage of the generic (minimal) points of \( \text{Spec } \tilde{B} \). So in general, \( \Gamma(Y) \subseteq \partial(Y) \), with equality when \( Y \) has dimension 1. Note that \( \Gamma(Y) \) is always finite, but \( \partial(Y) \) usually isn’t.

As an example, if \( Y \) is a strict closed annulus \( \{ r \leq |z| \leq R \} \), then \( \partial(Y) = \Gamma(Y) \) consists of 2 points: the Gauss point of the outer boundary and the Gauss point of the inner boundary. The \textit{skeleton} of the annulus is the unique path between these two points.

Unlike much of what we’ve been doing, the concepts of relative interior and relative boundary have no clear analogs in either scheme theory or classical rigid analysis.

## 2 Inner Homomorphisms

Inner homomorphisms are a generalization of the notion of relative interior and relative boundary. Let \( A \) be a \( k \)-affinoid algebra, \( B \) and \( C \) be Banach \( A \)-algebras with \( B \) an \( A \)-affinoid algebra, and let \( \phi : B \to C \) be a
bounded homomorphism of $A$-algebras. This yields diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{\phi} & M(C) \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M(C) & \xrightarrow{\phi} & M(B) \\
\downarrow & & \downarrow \\
M(A) & \xrightarrow{\phi} & M(\mathcal{A}) \\
\end{array}
\]

**Definition 3.** The morphism $\phi$ is *inner with respect to* $A$ if there exists an admissible surjection of Banach $A$-algebras

\[A(r_1^{-1}T_1, \ldots, r_n^{-1}T_n) \to B\]

sending $T_i \mapsto f_i$, such that $\|\phi(f_i)\|_{\text{sup}} < r_i$ for all $i$, where $\| \cdot \|_{\text{sup}}$ denotes the spectral norm on $C$, or equivalently, the supremum-norm on $M(C)$.

The case where $M(C)$ is a point corresponds to the previous notion of relative interior. Specifically, if $X = M(A) \to Y = M(B)$ is a morphism of $k$-affinoid spaces, then

\[\text{Int}(Y/X) = \{ y \in Y | \chi_y : B \to \mathcal{H}(y) \text{ is inner with respect to } A \}\]

We close by noting that if $M(C)$ and $M(B)$ are strict $k$-affinoids, then $\phi : B \to C$ is inner with respect to $M(k)$ if and only if Spec $C \to$ Spec $B$ is a constant map. We will observe this in the next lecture.
Suppose $A$ and $B$ are $k$-affinoid algebras. Recall that a bounded homomorphism $\phi : B \to C$ of Banach $A$-algebras is inner with respect to $A$ if there is an admissible surjection

$$A\langle r_1^{-1}T_1, \ldots, r_n^{-1}T_n \rangle \to B$$

sending $T_i \mapsto f_i$, such that $||\phi(f_i)||_{\text{sup}} < r_i$ for all $i$, where $|| \cdot ||_{\text{sup}}$ is the spectral norm (supremum-norm) on $C$.

One case of interest is when $\phi : B \to C = B$ is the identity map. Then $\phi$ is inner with respect to $A$ if and only if $A \to B$ is boundaryless, i.e., $\partial(M(B)/M(A)) = \emptyset$. This is a consequence of Proposition 2.5.9 in [Berk], which says that a map $M(C) \to M(B)$ is inner with respect to $M(A)$ if and only if the image is in the relative interior of $M(B)$ over $M(A)$.

Another special case of the notion of inner homomorphism is when $C$ is $\mathcal{H}(y)$ for some $y \in M(B)$ and $\phi$ is $\chi_y$, the character $B \to \mathcal{H}(y)$. Then $\chi_y$ is inner with respect to $A$ if and only if $y \in \text{Int}(M(A)/M(B))$. This follows from the definitions, since $\phi(f_i) = f_i(y)$ and $|| \cdot ||_{\text{sup}}$ on $\mathcal{H}(y)$ is the absolute value, so $||\phi(f_i)||_{\text{sup}} = |f_i(y)|$.

As a convention, “strict” will henceforth mean “strict and non-trivially valued.”

**Theorem 1.** Let $A$ and $B$ be strictly $k$-affinoid algebras. Let $\varphi : A \to B$ be a bounded homomorphism of affinoid algebras. Then the following are equivalent:

1. $\varphi$ is finite
2. $\tilde{\varphi}$ is finite
3. $\text{id}_B$ is inner with respect to $A$.

As noted above, the third condition is also equivalent to $\varphi$ being boundaryless. The equivalence of (1) and (2) is nontrivial, but proved in Theorem 6.3.5/1 of [BGR].

Berkovich generalized Theorem 1 to the following relative form, which is a component of Proposition 2.5.2 in [Berk]:

**Theorem 2.** (Berkovich) Let $A, B$ be strictly $k$-affinoid algebras, and $\phi : B \to C$ be a bounded homomorphism of $A$-algebras. Then $\phi$ is inner with respect to $A$ if and only if $\tilde{\phi}(\tilde{B})$ is finite over $\tilde{\phi}(\tilde{A})$ (as subrings of $\tilde{C}$).

Note that when $\phi$ is the identity map, this yields the equivalence of (2) and (3) in Theorem 1. Also note that $\phi(\tilde{B})$ is finite over $\tilde{\phi}(\tilde{A})$ if and only if it is integral, because the map $\tilde{\phi}(\tilde{A}) \to \tilde{\phi}(\tilde{B})$ is finite type.

---

1 It can be shown using the Banach open mapping theorem, that when $|k|$ is nontrivial, admissibility follows from surjectivity and boundedness.
We will prove Theorem 1 (assuming [BGR] Theorem 6.3.5/1), since the proof of Theorem 2 seems to be more difficult. We first state a lemma, which is [BGR 6.1.1/4] and [Berk 2.1.5] for the nonstrict case:

**Lemma 1.** If \( \varphi : A \to B \) is a bounded homomorphism of \( k \)-affinoid algebras and \( r_1, \ldots, r_n > 0 \), then given \( f_1, \ldots, f_n \in B \) with \( \| f_i \|_{\sup} \leq r_i \), there is a unique bounded homomorphism \( \Phi : A(r_1^{-1}T_1, \ldots, r_n^{-1}T_n) \to B \) such that \( \Phi(T_i) = f_i \) and \( \Phi|A = \varphi \). Conversely, if \( \Phi : A(r_1^{-1}T_1, \ldots, r_n^{-1}T_n) \to B \) is given and \( f_i = \Phi(T_i) \), then \( \| f_i \|_{\sup} \leq r_i \).

Given this, we can prove Theorem 1:

**Proof (of Theorem 1).** As noted above, the equivalence of (1) and (2) is Theorem 6.3.5/1 of [BGR]. We prove (3) \( \implies \) (2) and (1) \( \implies \) (3). For (3) \( \implies \) (2), suppose that \( \text{id}_B \) is inner with respect to \( A \). Then there is an admissible surjection \( \pi : A(r_1^{-1}T_1, \ldots, r_n^{-1}T_n) \to B \) with \( \| \pi(T_i) \|_{\sup} < r_i \) for each \( i \). Since \( \sqrt{|k^\times|} \) is dense, we can find \( \rho_i \in \sqrt{|k^\times|} \) with \( \| \pi(T_i) \|_{\sup} < \rho_i \leq r_i \). By Lemma 1, there is a map \( \pi' : A(\rho_1^{-1}T_1, \ldots, \rho_n^{-1}T_n) \to B \) sending \( T_i \) to \( \pi(T_i) \). Since \( \pi \) factors through \( \pi' \), \( \pi' \) is also surjective, and therefore admissible. Replacing \( \pi \) with \( \pi' \) and \( r_i \) with \( \rho_i \), we can assume without loss of generality that all the \( r_i \) live in \( \sqrt{|k^\times|} \).

Write \( r_i^\times = |c_i^{-1}| \) for some \( c_i \in k^\times \). Define \( \nu : A' = A(T_1, \ldots, T_n) \to A(r_1^{-1}T_1, \ldots, r_n^{-1}T_n) \) by \( \nu(T_i) = c_i T_i^\times \). This uniquely determines \( \nu \) by Lemma 1. Moreover, \( \nu \) is a finite map; this follows from the proof of Theorem 6.1.5/4 in [BGR].

Let \( \nu' = \pi \circ \nu : A' \to B \). The map \( \pi \) is finite because it is a surjection, so therefore \( \nu' \) is finite. By [BGR] 6.3.5/1, the map \( \nu' : A' = A[T_1, \ldots, T_n] \to B \) is finite. This implies that \( B \) is finite over \( \nu'(A') \). But \( \nu'(A') = \nu'(A) \) because \( \nu'(T_i) = 0 \) because \( \| \nu'(T_i) \|_{\sup} < 1 \) because \( \| \pi(T_i) \|_{\sup} < r_i \), thus \( A \to B \) is finite.

For the converse, we show that (1) implies (3), i.e., that if \( \varphi : A \to B \) is finite, then \( \text{id}_B \) is inner with respect to \( A \). Choose generators \( b_1, \ldots, b_n \) for \( B \) as an \( A \)-module. Rescaling by scalars, we can assume that every \( b_i \) has spectral norm at most 1, so that \( b_i \in B^\circ \). Now let \( c_i \in k \) be any scalar with \( |c_i| < 1 \). By Lemma 1 we can construct a map \( A(T_1, \ldots, T_n) \to B \) by sending \( T_i \to cb_i \). This map is clearly surjective, since the \( cb_i \) generate \( B \) as an \( A \)-module. Since \( \| cb_i \|_{\sup} < 1 \), we see that \( \text{id}_B \) is inner with respect to \( A \).

This argument is hard to generalize to a proof of Theorem 2. If \( \phi : B \to C \) is a bounded morphism of \( A \)-algebras, with \( C \) not necessarily \( k \)-affinoid, then we don’t know that \( \phi \) is finite if and only if \( \tilde{\phi} \) is. To generalize the implication (2) \( \implies \) (3), we therefore can’t use the intermediate step (2) \( \implies \) (1) \( \implies \) (3), since the generalization of (2) \( \implies \) (1) is probably false.

The rough idea of the proof is as follows. We need to show that if \( \tilde{\phi} : \tilde{A} \to \tilde{B} \) is finite, then there exists an admissible surjection \( \pi : A(T_1, \ldots, T_n) \to B \) with \( \| \pi(T_i) \|_{\sup} < 1 \). The idea is to choose a set of integral generators of \( B \) over \( A \), then lift them to \( B^\circ \). Also, lift their defining polynomial relators to polynomials over \( A^\circ \). This yields inequalities of the form \( \| p_i(b_i) \|_{\sup} < 1 \), where the \( b_i \)'s are the lifts of the generators and the \( p_i(X)'s \) are the polynomials over \( A^\circ \). For each such \( b_i \), we use \( 1 + \deg p \) of the \( T_i$’s, sending \( T_0 \to cb_1 \), \( T_1 \to p(b) \), \( T_2 \to bp(b) \) and so on up to \( T_n \to b^{n-1}p(b) \), where \( n = \deg p \). This map turns out to be surjective.

Details can be found in Lemma 2.5.4 of [Berk].
Let $k$ be a complete nonarchimedean valued field. Last time, we discussed inner homomorphisms. Recall (see Proposition 2.5.9 in Berkovich) that if $A$ and $B$ are affinoid $k$-algebras and $C$ is a Banach $A$-algebra, a bounded $A$-homomorphism $\phi : B \to C$ is inner with respect to $A$ if and only if the image of $\mathcal{M}(C)$ lies in $\text{Int}(\mathcal{M}(B)/\mathcal{M}(A))$. Furthermore, a bounded homomorphism $\phi$ of strictly $k$-affinoid algebras is finite if and only if the map $id_B$ is $A$-inner, if and only if for all $y \in Y = \mathcal{M}(B)$, we have that $\tilde{\chi}_y(B)$ is finite over $\tilde{\chi}_x(A)$, where $x = \phi(y)$. Also, we have $\partial(Y/X) = \emptyset$ if and only if $\phi$ is a finite morphism. Recall that $\text{Int}(Y/X) = \{y \in Y : \chi_y : B \to \mathcal{H}(y) \text{ is } A\text{-inner}\}$. So $id_B$ is $A$-inner if and only if for all $y \in \mathcal{M}(B)$, we have that $\chi_y : B \to \mathcal{H}(y)$ is inner.

Let $Y = \mathcal{M}(B)$ be a strict $k$-affinoid space. We defined the absolute interior of $Y$ to be $\text{Int}(Y) = \text{Int}(\mathcal{M}(B)/\mathcal{M}(k))$. Let $\pi : \mathcal{M}(B) \to \text{Spec}(\tilde{B})$ be the canonical reduction map.

**Proposition 1.** $\text{Int}(Y) = \pi^{-1}(\text{Max}(\tilde{B}))$.

**Proof.** Given $y \in Y$, we have $y \in \text{Int}(Y)$ if and only if $\tilde{\chi}_y(B)$ is finite over $\tilde{\chi}_x(A) = \tilde{k}$; here $A = k$. This happens if and only if $\tilde{\chi}_y(B)$ is a field. Note that $\tilde{\chi}_y(B) = \tilde{B}/\ker(\tilde{\chi}_y)$, so $\tilde{\chi}_y(B)$ is a field if and only if $\pi(y)$ is a maximal ideal. \qed

**Functorial properties of $\text{Int}(Y/X)$** (see Proposition 2.5.8 in Berkovich).

1. Suppose we have a cartesian diagram of affinoids

\[
\begin{array}{ccc}
Y' & \longrightarrow & X' \\
\downarrow{\psi'} & & \downarrow{} \\
Y & \longrightarrow & X
\end{array}
\]

obtained via the completed tensor product. That is, if $Y = \mathcal{M}(B)$, $X = \mathcal{M}(A)$, and $X' = \mathcal{M}(A')$, then $Y' = \mathcal{M}(B \otimes_A A')$ is the fiber product. Then

\[ (\psi')^{-1}(\text{Int}(Y/X)) \subseteq \text{Int}(Y'/X'). \]

2. Let $K$ be an extension of $k$, and let $\psi : Y \otimes_k K \to Y$ be the base change map. Then

\[ \psi^{-1}(\text{Int}(Y/X)) \subseteq \text{Int}(Y \otimes K/X \otimes K). \]

3. Given a composition $Z \xrightarrow{\phi} Y \xrightarrow{\psi} X$, we have

\[ \text{Int}(Z/X) = \text{Int}(Z/Y) \cap \psi^{-1}(\text{Int}(Y/X)). \]
Next, we turn to Temkin’s proof of the Gerritzen-Grauert theorem. Let \( X = \mathcal{M}(A) \) be a strictly \( k \)-affinoid space.

**Theorem 2** (Gerritzen-Grauert). Every strict affinoid subdomain in \( X \) is a finite union of strict rational domains.

**Proof sketch.** Let \( Y = \mathcal{M}(B) \) be an affinoid subdomain of \( X = \mathcal{M}(A) \) (all affinoid and rational domains are strict for the rest of today). Our basic strategy is to prove the following:

\[(\ast) \quad \text{For each } y \in Y, \text{ there exists a rational domain } X' \subset X \text{ such that}
\]

1. \( Y' = X' \cap Y \) is a neighborhood of \( y \) in \( Y \), and
2. \( \partial(Y'/X') = \emptyset \).

Assuming \((\ast)\), we claim that \( Y' \) is a rational subdomain of \( X \). Given the claim, for each \( y \in Y \), we get a rational domain \( Y'_y \subset Y \) satisfying conditions (1) and (2), and the collection \( \{Y'_y\} \) covers \( Y \). Since \( Y \) is compact, \( \{Y'_y\} \) has a finite subcover \( Y = Y'_{y_1} \cup \cdots Y'_{y_m} \), and we are done.

We follow a suggestion of Ducros to prove the claim. Let \( Y' = \mathcal{M}(B') \) and \( X' = \mathcal{M}(A') \). Since the map \( Y' \to X' \) is a boundaryless map of affinoids by (2), it is a finite map (Berkovich 2.5.13). Also, since \( Y' \) is an affinoid subdomain of \( X' \) (for the intersection of an affinoid domain and rational domain is affinoid), the map \( Y' \to X' \) is flat by 7.3.2/6 in [BGR]. In other words, we have a ring map \( A' \to B' \) which is finite and flat. Then the scheme map \( \text{Spec}(B') \to \text{Spec}(A') \) is both open (by flatness) and closed (by finiteness), hence the image of \( \text{Spec}(B') \) in \( \text{Spec}(A') \) is a union of connected components in the Zariski topology. Thus, there exists an idempotent \( e' \in A' \) such that \( Y' = \{p \in X' : e'(p) = 0\} \). (Thus \( e' = 0 \) on the components corresponding to \( Y' \), and \( e' = 1 \) on the other components). Rewriting, \( Y' = \{p \in X' : |e'(p)| \leq \frac{1}{2}\} \) is a Weierstrass, and hence rational, subdomain of \( X' \subset X \). Since “rational subdomain” is transitive, we conclude that \( Y' \) is a rational subdomain of \( X \).
Harmonic functions on $\mathcal{M}(\mathbb{Z})$ and global $k$-analytic spaces

Michael Daub

April 17, 2012

1 Harmonic functions on $\mathcal{M}(\mathbb{Z})$

We conclude the discussion on harmonic functions from last time by describing a class of harmonic functions on $\mathcal{M}(\mathbb{Z})$. First, let us set some notation. We shall denote by $(1,\varepsilon)$ the point of $\mathcal{M}(\mathbb{Z})$ corresponding to the seminorm $\cdot_1^\varepsilon$, where $\cdot_1$ is the usual Archimedean absolute value on $\mathbb{Z}$ and $0 \leq \varepsilon \leq 1$. For any prime $p$, write $(p,\varepsilon)$ for the point of $\mathcal{M}(\mathbb{Z})$ corresponding to the seminorm $\cdot_p^\varepsilon$, where $\cdot_p$ is the $p$-adic absolute value on $\mathbb{Z}$ satisfying $|p|_p = 1/p$ and $0 \leq \varepsilon \leq \infty$. Here $\cdot_p^\varepsilon$ is the seminorm on $\mathbb{Z}$ such that $|p|^\varepsilon = 0$ and $|n|^p_\varepsilon = 1$ when $p \nmid n$. Note that when $\varepsilon = 0$, all of these norms coincide.

Let $f \in \mathbb{Q}^\times$. Then we can define a function $\log |f| : \mathcal{M}(\mathbb{Z}) \to \mathbb{R} \cup \{\pm \infty\}$ by $(v,\varepsilon) \mapsto \log |n|^v_\varepsilon$. Writing $f = \frac{a}{b}$ with $\gcd(a,b) = 1$, then $\log |f|$ defines a map from $U$ to $\mathbb{R}$, where $U = \mathcal{M}(\mathbb{Z}) \setminus \{(p,\infty) : p \mid ab\}$.

**Proposition 1.** $\log |f|$ is harmonic on $U$.

**Proof.** For any fixed place $v$ of $\mathbb{Z}$, on the branch $(v,\varepsilon)$ the function takes the form $\log |n|^v_\varepsilon = \varepsilon \log |n|_v$. Hence, the function is linear on each branch of $\mathcal{M}(\mathbb{Z})$. So it suffices to verify that $\log |n|$ is harmonic at the trivial seminorm. The slope on the branch $(\infty,\varepsilon)$ is equal to $\log |n|_1$, while on any branch $(p,\varepsilon)$ the slope is $\log |n|_p$. Thus, by the product formula, the sums of the slopes on all branches of $\cdot_0$ is

$$\sum_v \log |n|_v = \log \left( \prod_v |n|_v \right) = \log(1) = 0.$$ 

$\square$

2 Global $k$-analytic spaces

In this section, $X$ will be a locally Hausdorff topological space. We start off with a few definitions.

**Definition 1.** A *quasinet* on $X$ is a collection $\tau$ of compact Hausdorff subsets $V \subseteq X$ such that each $x \in X$ has a neighborhood that is a finite union of elements of $\tau$. A quasinet is called a *net* if for all $V,V' \in \tau$, $\{W \in \tau : W \subseteq V \cap V'\}$ is a quasinet on $V \cap V'$.

**Definition 2.** A *$k$-affinoid atlas* on $(X,\tau)$ is an assignment of a $k$-affinoid algebra $A_V$ to each $V \in \tau$ along with a homeomorphism $V \longrightarrow \mathcal{M}(A_V)$, which is functorial in the sense that if $V,V' \in \tau$ and $V' \subseteq V$, then there is a bounded homomorphism of $k$-affinoid algebras $A_V \longrightarrow A_{V'}$, identifying $(V',A_{V'})$ with an affinoid subdomain of $(V,A_V)$. A *$k$-analytic space* is a triple $(X,\mathcal{A},\tau)$ consisting of a topological space $X$, a $k$-affinoid atlas $\mathcal{A}$, and a net $\tau$. 

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Remark. If all $A_V$ are strict $k$-affinoid algebras, then $(X, A, \tau)$ is called a strict $k$-analytic space.

We conclude with a few examples.

**Example 1.** Let $X = (\mathbb{A}^1)^{\text{an}}$, the multiplicative seminorms on $k[T]$ extending the absolute value on $k$. For every $r > 0$, let $X_r = \{ x \in X : |T|_x \leq r \}$. Restriction to $k[T]$ induces a homeomorphism $i_r : \mathcal{M}(k\langle r^{-1}T \rangle) \sim X_r$.

Let $\tau = \{ X_r \}_{r > 0}$. This forms a net on $X$, and $X_r \mapsto k\langle r^{-1}T \rangle$ is a $k$-affinoid atlas.

**Example 2.** A $k$-affinoid atlas on $X = (\mathbb{P}^1)^{\text{an}}$ is given by $V_1 = \mathcal{B}(0,1)$, $V_2 = X \setminus \mathcal{B}(0,1)^c \simeq \mathcal{B}(0,1)$, and $V_3 = V_1 \cap V_2 = \mathcal{B}(0,1) \setminus \mathcal{B}(0,1)^c$. Notice that $V_1 \cup V_2$ is a neighborhood in $X$ of the Gauss point $\xi_{\text{Gauss}}$, but no $V_i$ is itself. This justifies the inclusion of finite unions in the definition of quasinet instead of requiring each point to have a neighborhood in $\tau$. Of course, there is a net $\tau$ on $(\mathbb{P}^1)^{\text{an}}$ such that every point has a neighborhood in $\tau$, but the definition we are using allows for the use of more convenient nets, such as the one in this example.

**Example 3.** Let $A$ be a $k$-affinoid algebra, and set $X = \mathcal{M}(A)$. Then we can put many different nets on $X$. In particular, $\tau = \{ X \}$ is a net with atlas $X \mapsto A$, and $\tau' = \{ V \subseteq A : V \text{ affinoid subdomain} \}$ is a net with atlas $V \mapsto A_V$. We would like to form a category of $k$-analytic spaces in which $(X, A, \tau)$ and $(X, A', \tau')$ are isomorphic. There is an obvious definition of morphism of $k$-analytic spaces, and next time we will see that by inverting a class of these morphisms, we achieve the desired category.
Morphisms of $k$-analytic spaces and localizations of categories

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1 Morphisms of $k$-analytic spaces

Recall from last time that a $k$-analytic space is a triple $(X, \mathcal{A}, \tau)$ where $X$ is a locally Hausdorff topological space, $\tau$ is a net on $X$, and $\mathcal{A}$ is a $k$-affinoid atlas. That is, $\tau$ is a quasinet, i.e. a collection $\{V_i\}$ of compact Hausdorff subsets of $X$ such that every $x \in X$ has a neighborhood of the form $V_i \cup \cdots \cup V_n$. Furthermore, $\tau$ is a net, meaning that for all $V_i, V_j \in \tau$, $\{W \in \tau : W \subseteq V_i \cap V_j\}$ is a quasinet on $V_i \cap V_j$. Finally, $\mathcal{A}$ is a $k$-affinoid atlas, associating a $k$-affinoid algebra $A_V$ for each $V \in \tau$, a homeomorphism $V \tilde{\to} \mathcal{M}(A_V)$, and for each $V' \subseteq V$ a bounded homomorphism $A_V \rightarrow A_{V'}$.

Example 1. Let $D \subseteq \mathbb{R}^2$ be the unit disk. Consider the collection $\tau = \{\sigma_j\}_{1 \leq j \leq 6}$, where $\sigma_j$ is the closed sector bounded by the line segments connecting the origin to $e^{2\pi ij/6}$. Then $\tau$ is a quasinet, but not a net. Indeed, the collection $\{W \in \tau : W \subseteq \sigma_j \cap \sigma_k\}$ is empty for any $j \neq k$. If we let $\rho_1 = \sigma_1 \cap \sigma_6$ and $\rho_j = \sigma_j \cap \sigma_{j-1}$ for $2 \leq j \leq 6$, then the collection $\{\sigma_j, \rho_k, 0\}_{1 \leq j, k \leq 6}$ forms a net.

Definition 1. A strong morphism $\varphi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau')$ of $k$-analytic spaces is a continuous map $\varphi : X \rightarrow X'$ such that for every $V \in \tau$ there is a $V' \in \tau'$ with $\varphi(V) \subseteq V'$, together with a compatible system of morphisms of $k$-affinoid spaces

$$\varphi_{V/V'} : (V, A_V) \rightarrow (V', A_{V'})$$

for all $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subseteq V'$.

Define $\tilde{k}$-An as the category with triples $(X, \mathcal{A}, \tau)$ as objects and strong morphisms of $k$-analytic spaces as morphisms. A strong morphism is called a quasi-isomorphism if $\varphi : X \tilde{\to} X'$ is a homeomorphism and each $\varphi_{V/V'}$ identifies $V$ with an affinoid subdomain of $V'$. The category $k$-An of $k$-analytic spaces is the localization of $\tilde{k}$-An with respect to the quasi-isomorphisms.

2 Localizations of categories

In this brief digression from $k$-analytic spaces, set-theoretic issues will be ignored, as they will not cause problems in the case we are interested in. Let $\mathcal{C}$ be a category, and $S$ a class of morphisms. We want to define a new category $\mathcal{C}[S^{-1}]$ having the following universal property: there is a functor $Q : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$ satisfying

1. $Q(s)$ is an isomorphism for all $s \in S$.
2. Any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors uniquely through $Q$. 

1
Although we will not prove it here, it is true that localizations always exist. If $S$ is particularly nice, then $\mathcal{C}[S^{-1}]$ admits a concrete description. Namely, if $S$ admits a "calculus of right fractions" in the sense of [GZ], then

- $\text{Ob}(\mathcal{C}[S^{-1}]) = \text{Ob}(\mathcal{C})$, and

- $\text{Mor}(A, B)$ consists of equivalence classes of diagrams

$$A \xleftarrow{s} A' \xrightarrow{f} B$$

with $s \in S$ and $f \in \text{Mor}(A', B)$.

Furthermore, if $f, g \in \text{Mor}(A, B)$, then $Q(f) = Q(g)$ if and only if there exists $s \in S$ such that $f \circ s = g \circ s$.

It is not hard to verify that $\widetilde{k-\text{An}}$ with $S$ the quasi-isomorphisms admits a calculus of right fractions. A much more difficult fact due to Temkin is that the category of strictly $k$-analytic spaces is a full subcategory of $k-\text{An}$.

References

Math 274: Non-Archimedean Geometry

Andrew Niles

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“Good” spaces

**Definition 1.** A $k$-analytic space $X$ is **good** if every point $x \in X$ has an affinoid neighborhood. (Recall in general that $x$ only has a neighborhood which is a finite union of affinoid subdomains.)

**Remark 1.**
1. In Berkovich’s red book, he defined a category of analytic spaces over $k$ which he later generalized (in his IHÉS paper). The spaces from his book turn out to be exactly the good spaces.
2. There are natural analytification functors

\[
\begin{align*}
\text{Schemes of finite type over } k & \quad \longrightarrow \quad k\text{-An} \\
\text{Quasi-separated, quasi-paracompact rigid } k\text{-analytic spaces} & \quad \longrightarrow \quad k\text{-An}.
\end{align*}
\]

If $X$ is a scheme (of finite type over $k$), then $X^{\text{an}}$ is good. But if $X$ is a rigid space, $X^{\text{an}}$ is not necessarily good.

**Example 1.** Let $\mathbb{B}(0, 1) = \mathcal{M}(k(S, T)) \subseteq (\mathbb{A}^2)^{\text{an}}$ be the closed unit polydisk, and let

\[\mathbb{B}(0, 1)^- = \{ x \in \mathbb{B}(0, 1) : |S|_x < 1, |T|_x < 1 \}\]

be the open unit polydisk. Let

\[
X = \mathbb{B}(0, 1) \setminus \mathbb{B}(0, 1)^-
= \{ x \in \mathbb{B}(0, 1) : |S|_x = 1 \text{ or } |T|_x = 1 \}
= \{ x \in \mathbb{B}(0, 1) : |S|_x = 1 \} \cup \{ x \in \mathbb{B}(0, 1) : |T|_x = 1 \}.
\]

It turns out that $X$ is not good. (We will see this using Temkin’s criterion. The idea is that the “reduction" of $X$ is $\mathbb{A}^2_k \setminus \{0\}$, which is not an affine scheme.)

**Analytic domains**

**Definition 2.** A subset $V \subseteq X$ is a **$k$-analytic domain** if there is a covering $\{V_i\}$ of $V$ with each $V_i$ $k$-affinoid, such that $\{V_i\}$ is a quasi-net on $V$. (This is the analogue of an “admissible open" for the strong G-topology in rigid analysis.)
• A covering of an analytic domain $V$ by analytic domains $V_i$ is *admissible* if $\{V_i\}$ is a quasi-net on $V$. (This is the analogue of an “admissible covering by admissible opens” in rigid analysis.)

**Remark 2.** Every open subset of $X$ is an analytic domain in a natural way, as is any finite union of affinoid domains.

**Remark 3.** The category $k$-$\text{An}$ admits fiber products. These are constructed similarly to fiber products of schemes. The category of $k$-affinoid spaces is a full subcategory of $k$-$\text{An}$, and for $X = \mathcal{M}(A), Y = \mathcal{M}(B), Z = \mathcal{M}(C)$, and morphisms $Y \to X, Z \to X$, the fiber product is

$$Y \times_X Z := \mathcal{M}(B \widehat{\otimes}_A C).$$

**Morphisms**

**Definition 3.** A morphism $f : Y \to X$ of $k$-analytic spaces is a *closed immersion* (resp. *finite*) if there is an admissible covering $\{X_i\}$ of $X$ by $k$-affinoid domains such that $Y_i = f^{-1}(X_i) = X_i \times_X Y$ is $k$-affinoid for all $i$, and the corresponding homomorphisms of $k$-Banach algebras $A_{X_i} \to A_{Y_i}$ are surjective (resp. finite) and admissible. (If $|k^*| \neq 1$, then admissibility is automatic by the Banach open mapping theorem.)

**Exercise 1.** The class of closed immersions (resp. finite morphisms) is closed under composition, and any base change of a closed immersion (resp. finite morphism) is a closed immersion (resp. finite).

**Definition 4.** Let $f : Y \to X$ be a morphism of $k$-analytic spaces. The *relative interior* of $f$ is the set $\text{Int}(Y/X)$ of all $y \in Y$ such that, for any affinoid $U \subseteq X$ containing $x = f(y)$, there is an affinoid $V \subseteq f^{-1}(U)$ which is a neighborhood of $y$ in $f^{-1}(U)$, and $y \in \text{Int}(V/U)$ as previously defined for affinoids.

The *relative boundary* of $f$ is $\partial(Y/X) := Y \setminus \text{Int}(Y/X)$.

We say $f$ is *boundaryless* if $\partial(Y/X) = \emptyset$. (Berkovich would say $f$ is “closed”.)

**Remark 4.** If $\partial(X) = \emptyset$, then $X$ is good.

**Definition 5.** A morphism $f : Y \to X$ of $k$-analytic spaces is *proper* if it is boundaryless and topologically proper (i.e. the inverse image of every compact subset of $X$ is compact).
1 Properties of morphisms

Definition 1. Recall that a morphism \( f : Y \to X \) of \( k \)-analytic spaces is a closed immersion (resp. finite) if there exists an admissible covering of \( X \) by affinoid domains such that \( Y_i = X_i \times_X Y \) is a \( k \)-affinoid and \( \mathcal{A}_{X_i} \to \mathcal{A}_Y \) are surjective (resp. finite) and admissible. Note that the admissible condition is automatic if \( k \) is non-trivially valued.

Definition 2. A morphism \( f : Y \to X \) is separated if \( \Delta : Y \to Y \times_X Y \) is a closed immersion.

Definition 3. A morphism \( f : Y \to X \) is boundaryless if \( \partial(Y/X) = 0 \), where \( \partial(Y/X) = Y \setminus \text{Int}(Y/X) \), and \( \text{Int}(Y/X) \) is defined by the set of points \( y \in Y \) such that for any affinoid domain \( U \subset X \) containing \( x = f(y) \), there exists an affinoid neighborhood \( V \subset f^{-1}(U) \) of \( y \) such that \( y \in \text{Int}(V/U) \).

Remark 1. If \( Y \) has no absolute boundary (i.e., \( \partial(Y/k) = \emptyset \)), then \( Y \) is good.

Definition 4. A morphism \( f : Y \to X \) is proper if \( \partial(Y/X) = \emptyset \) and \( f \) is proper as a map of topological spaces.

Facts 1. 1) Finite morphisms are proper.
2) More generally, \( f : Y \to X \) is finite if and only if it is proper and has finite fibers.
3) Properness implies separatedness.

Lemma 1. (Transitivity property of the relative interior) If \( Z \overset{\psi}{\to} Y \overset{\phi}{\to} X \) are morphisms of \( k \)-analytic spaces, then
\[
\text{Int}(Z/X) = \text{Int}(Z/Y) \cap \phi^{-1}(\text{Int}(Y/X)).
\]

2 Application of the relative interior

Let \( k \) be a non-trivially valued field, \( \mathcal{A} \) a strict \( k \)-affinoid algebra, \( X = \text{Sp}(\mathcal{A}) \), \( Z = \text{Sp}(\mathcal{A}/I) \) where \( I = (f_1, \ldots, f_n) \) is an ideal in \( \mathcal{A} \), which is automatically closed. Suppose that \( Z \subset U \subset X \) where \( U \) is a finite union of affinoid subdomains of \( X \).

Theorem 1. There exists \( \epsilon > 0 \) such that
\[
\{ x \in X : |f_i(x)| \leq \epsilon, \ldots, |f_n(x)| \leq \epsilon \} \subset U.
\]

Proof. Since \( X^{\text{an}} = \mathcal{M}(\mathcal{A}) \) is compact and Hausdorff, it suffices to prove that \( U^{\text{an}} \) is a neighborhood of \( Z^{\text{an}} \) in \( X^{\text{an}} \). Since \( U^{\text{an}} \) is a \( k \)-analytic domain in \( X^{\text{an}} \), \( (U^{\text{an}})^\circ = \text{Int}(U^{\text{an}}/X^{\text{an}}) \) where \( (U^{\text{an}})^\circ \) is the topological
interior of $U^{\text{an}}$ in $X^{\text{an}}$. Since $Z^{\text{an}} \to X^{\text{an}}$ is a closed immersion, it is finite and therefore boundaryless. Thus, $Z^{\text{an}} = \text{Int}(Z^{\text{an}}/X^{\text{an}})$. Now, by the transitivity property of the relative interior,

$$Z^{\text{an}} = \text{Int}(Z^{\text{an}}/X^{\text{an}}) = \text{Int}(Z^{\text{an}}/U^{\text{an}}) \cap \text{Int}(U^{\text{an}}/X^{\text{an}}) \subseteq \text{Int}(U^{\text{an}}/X^{\text{an}}) = (U^{\text{an}})^\circ.$$  

\section{Analytification}

We want to define a functor $X \mapsto X^{\text{an}}$ from the category of schemes of finite type over $k$ to $k$-analytic spaces.

**Affine Case.** If $X = \text{Spec} A$, where $A = k[x_1, \ldots, x_n]/I$ for an ideal $I$, let $X^{\text{an}}$ be the set of multiplicative seminorms on $A$ extending $| \cdot |$ on $k$ with the weakest topology such that $x \mapsto |f|_x$ is continuous for all $f \in A$. Let $\tau = \{X_r\}_{r>0}$, where $X_r = \mathcal{M}(k\langle r^{-1}x_1, \ldots, r^{-1}x_n\rangle/I)$, $A_{X_r} = k\langle r^{-1}x_1, \ldots, r^{-1}x_n\rangle/I$.

**General Case.** If $X$ is a scheme of finite type over $k$, let $X^{\text{an}}$ be the set of $x = (\zeta, | \cdot |)$ such that $\zeta \in X$ and $| \cdot |_\zeta$ is an extension of $| \cdot |$ on $k$ to $k(\zeta)$. Note that this definition coincides with the definition in the affine case. The topology is the weakest one for which $U^{\text{an}} \subseteq X^{\text{an}}$ is open for all open affine subsets $U \subseteq X$ and $x \mapsto |f(x)|$, $U^{\text{an}} \to \mathbb{R}$ is continuous for all $f \in \mathcal{O}_X(U)$.

**Next time.** We will discuss the universal property of analytification.
1 Analytification of Schemes

Let $k$ be a non-archimedean, complete field. We are interested in a functor

$\{\text{schemes of finite type}/k\} \longrightarrow \{\text{good strictly } k\text{-analytic spaces}\}$

sending $X$ to $X^{an}$. This represents the functor

$\{\text{good strictly } k\text{-analytic spaces}\} \longrightarrow \text{Sets}$

defined by

$Y \longmapsto \{\text{morphisms of locally ringed spaces } (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)\}$. 

In particular, one gets a canonical morphism of locally ringed spaces

$(X^{an}, \mathcal{O}_{X^{an}}) \rightarrow (X, \mathcal{O}_X)$. 

Let $X$ be a $k$-analytic space.

Fact 1. The association $V \mapsto \mathcal{A}_V$ (where $V$ is an affinoid domain in $X$) extends uniquely to a $\mathcal{G}$-sheaf $\mathcal{O}_{X^{\mathcal{G}}}$ mapping $V \mapsto \mathcal{A}_V$ with respect to the strong $\mathcal{G}$-topology

$\{\text{k-analytic domains, admissible coverings}\}$. 

The restriction of $\mathcal{O}_{X^{\mathcal{G}}}$ to the open subsets $U \subset X$ is denoted $\mathcal{O}_X$. It is a sheaf in the usual (topological) sense.

In practice, one only makes use of $\mathcal{O}_X$ when $X$ is a good space.

Fact 2. If $X$ is good at $x$, then $\mathcal{H}(x)$ (computed with respect to any affinoid domain containing $x$) is canonically isomorphic to the completion of $\mathcal{K}(x)$, where $\mathcal{K}(x)$ is the residue field of the local ring $\mathcal{O}_{X,x}$.

This can fail for non-good spaces.

There is also an analytification functor

$\{\text{quasicompact, quasiseparated rigid spaces}/k\} \rightarrow \{\text{compact, Hausdorff, strictly } k\text{-analytic spaces}\}$

which “locally” sends $\text{Sp}(\mathcal{A})$ to $\mathcal{M}(\mathcal{A})$.

Berkovich shows in his IHES paper that this is an equivalence of categories.
Remark 1. In this setting, a rigid space is \textit{quasicompact} if it is an admissible finite union of affinoids (where admissible is automatic if the space is quasiseparated). A rigid space is \textit{quasiseparated} if the intersection of affinoids is a finite admissible union of affinoids.

More generally, assuming a local finiteness condition on both sides (instead of quasicompactness), one gets an equivalence of categories

\[ \{ \text{quasiseparated rigid spaces of finite type} \} \rightarrow \{ \text{paracompact Hausdorff strictly } k\text{-analytic spaces} \} \]

Remark 2. Recall that \textit{finite type} means that there exists a locally finite admissible covering by affinoid subdomains, and \textit{paracompact} means that every open cover has a locally finite subcover.

We have functors from \{\text{separated schemes of finite type}/k\} to each of the above categories. Adding it to the diagram gives a commutative diagram of functors:

\[ \begin{array}{ccc}
\{ \text{quasiseparated rigid spaces of finite type} \} & \rightarrow & \{ \text{paracompact Hausdorff strictly } k\text{-analytic spaces} \} \\
\uparrow & & \downarrow \\
\{ \text{separated schemes of finite type}/k \} 
\end{array} \]

2 \hspace{1em} \textbf{Properties of } \text{Sch} \rightarrow k\text{-An}

(1) A morphism \( f : Y \rightarrow X \) between schemes of finite type over \( k \) is a closed immersion (respectively finite, proper, separated, an isomorphism, etc.) if and only if \( f^{an} : Y^{an} \rightarrow X^{an} \) is.

(2) (A GAGA-type fact.) If \( X \) is proper then analytification induces an equivalence of categories

\[ \text{Coh}(X, \mathcal{O}_X) \rightarrow \text{Coh}(X^{an}, \mathcal{O}_{X^{an}}), \]

where Coh denotes the category of coherent sheaves (on the given locally ringed space). The functor \( X \rightarrow X^{an} \) is fully faithful on proper schemes.

Remark 3. For a locally ringed space \((Y, \mathcal{O}_Y)\), an \( \mathcal{O}_Y \)-module is \textit{coherent} if it is locally isomorphic to the cokernel of a morphism of locally free \( \mathcal{O}_Y \)-modules of finite rank. (This definition is equivalent to the characterization found in such texts as Hartshorne’s Algebraic Geometry.)

(3) – \( X^{an} \) is Hausdorff if and only if \( X \) is separated.
– \( X^{an} \) is compact if and only if \( X \) is proper.
– \( X^{an} \) is (path) connected if and only if \( X \) is connected. (\( X^{an} \) connected implies that \( X^{an} \) path-connected.)
1 Germs of analytic spaces and Riemann-Zariski spaces

We let $\text{Germs}_k$ denote the localization of the category of pointed strictly $k$-analytic spaces $(X, x)$ with respect to the morphisms $\varphi : (X, x) \to (Y, y)$ which induce an isomorphism from $X$ to a neighborhood of $y$.

Henceforth, when we write $(X, x)$, we mean its image in this localization, which we call the germ of $X$ at $x$.

**Definition 1.** A germ $(X, x)$ is good if $x$ has an affinoid neighborhood.

Let $\bar{k}$ be a field (for the moment, it is not necessary to assume that $\bar{k}$ is the residue field of $k$). Let $K/\bar{k}$ be a field extension. We will define an associated Riemann-Zariski space associated to $K/\bar{k}$ as follows.

As a set, $RZ_{\bar{k}}(K) = \{\text{valuation rings } \bar{k} \subset \mathcal{O} \subset K \text{ with fraction field } K\}$. We give $RZ_{\bar{k}}(K)$ the weakest topology for which $\{\mathcal{O} \mid f \in \mathcal{O}\}$ is open for every $f \in K$.

**Fact 1.** (Zariski) $RZ_{\bar{k}}(K)$ is a quasicompact and quasiseparated topological space.

We also define a category $\text{bir}_{\bar{k}}$ of birational spaces over $\bar{k}$. The objects of this category are triples $(X, K, \phi)$, where:

- $X$ is a connected, quasicompact and quasiseparated topological space
- $K$ is a field extension of $\bar{k}$
- $\phi : X \to RZ_{\bar{k}}(K)$ is a continuous map which is a local homeomorphism, i.e. for all $x \in X$, $\phi$ induces a homeomorphism between an open neighborhood of $x$ and an open subset of $RZ_{\bar{k}}(K)$.

The morphisms $(X, K, \phi) \to (Y, L, \psi)$ in $\text{bir}_{\bar{k}}$ are pairs $(h, i)$, where $h : X \to Y$ is continuous, $i : L \to K$ is a $k$-homomorphism, and the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow{\phi} & & \downarrow{\psi} \\
RZ_{\bar{k}}(K) & \xrightarrow{i^\#} & RZ_{\bar{k}}(L)
\end{array}
\]
2 Classical Motivation

Let \( \text{Var}_{\tilde{k}} \) be the category of pointed \( \tilde{k} \)-varieties (here a variety is an integral scheme of finite type), whose objects are maps \( \text{Spec}(K) \to Z \), where \( K/\tilde{k} \) is a field extension, \( Z \) is a variety over \( \tilde{k} \), and morphisms are commutative diagrams:

\[
\begin{array}{ccc}
\text{Spec}(K') & \longrightarrow & Z' \\
\downarrow f_{\eta} & & \downarrow f_s \\
\text{Spec}(K) & \longrightarrow & Z
\end{array}
\]

Let \( \text{Bir}_{\tilde{k}} \) be the localization of this category with respect to the family of maps:

\[
\mathcal{B} := \{ (f_{\eta}, f_s) \mid f_{\eta} \text{ is an isomorphism and } f_s \text{ is proper} \}
\]

Fact 2. \( \mathcal{B} \) admits a calculus of right fractions.

For any \( f : \text{Spec}(K) \to X \) in \( \text{Var}_{\tilde{k}} \), we can construct a birational space over \( \tilde{k} \) consisting of all pairs \( (\mathcal{O}, g) \), with \( \mathcal{O} \in \text{RZ}_{\tilde{k}}(K) \), and \( g : \text{Spec}(\mathcal{O}) \to X \) a morphism such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{f} & X \\
\downarrow & & \downarrow g \\
\text{Spec}(\mathcal{O}) & &
\end{array}
\]

(note that when \( X \) is separated, \( g \) is unique)

Fact 3. The above map defines a functor \( \text{Var}_{\tilde{k}} \to \text{bir}_{\tilde{k}} \) inducing an equivalence of categories \( \text{Bir}_{\tilde{k}} \to \text{bir}_{\tilde{k}} \).

3 Reduction of an analytic germ

Now, assume that \( \tilde{k} \) is the residue field of \( k \). We sketch the definition of the reduction functor \( \text{Germs}_{\tilde{k}} \to \text{bir}_{\tilde{k}} \).

Let \( X \) be a paracompact, Hausdorff, strictly \( k \)-analytic space, and \( X^{ad} \) the corresponding adic space. In terms of earlier lectures, we can interpret \( X^{ad} \) (as a topological space) as the space of prime filters of special subdomains of \( X \). Then there is a retraction map \( r : X^{ad} \to X \).

Take a point \( x \in X \). For any \( \nu \in r^{-1}(x) \), which can be regarded as a Krull valuation on \( \mathcal{H}(x) \) with valuation ring \( \mathcal{O}_\nu \), we have \( k^\nu \subset \mathcal{O}_\nu \subset \mathcal{H}(x)^\nu \), so it makes sense to consider the reduction \( \mathcal{O}_\nu \) (which is a valuation ring over \( \tilde{k} \) with fraction field \( \mathcal{H}(x) \)).

Definition 2. We define the reduction \( \overset{\sim}{(X, x)} \) of a germ \( (X, x) \) to be the birational space \( (\overset{\sim}{X}, K, \phi) \), where \( \overset{\sim}{X} = r^{-1}(x) \), \( K = \overset{\sim}{\mathcal{H}}(x) \), and \( \phi : \overset{\sim}{X} \to \text{RZ}_{\tilde{k}}(K) \) maps \( \nu \) to \( \mathcal{O}_\nu \).

4 Germ subdomains

Definition 3. A subdomain of a germ \( (X, x) \) is an equivalence class of pairs \( (Y, x) \) with \( Y \subset X \) an analytic subdomain containing \( x \), where \( (Y, x) \sim (Y', x) \) if \( Y \cap Y' \) is a neighborhood of \( x \) in both \( Y \) and \( Y' \).
Theorem 1. (Temkin) The reduction functor \( \text{Germs}_k \to \text{bir}_k \) establishes a bijection between germ subdomains of \((X, x)\) and quasicompact open subsets of the space \( \text{X} \) associated to \((X, x)\). This bijection preserves inclusions, finite unions, and intersections.

Definition 4. A birational space \((\text{X}, K)\) is affine if \( \phi \) is injective and its image is of the form \( RZ_k(K)[f_1, \ldots, f_n] := \{ \mathcal{O} \in RZ_k(K) \mid f_i \in \mathcal{O} \text{ for all } i \} \) for some \( f_i \in K \).

Theorem 2. The germ \((X, x)\) is good if and only if \((\text{X}, x)\) is affine.

Definition 5. A morphism \( Y \to X \) of analytic spaces is boundaryless (or inner) at \( y \in Y \) if \( y \in \text{Int}(Y/X) \).

A map \( \mathcal{f} : \text{Y} \to \text{X} \) of birational spaces over \( k \) is separated (resp. proper) if \( \mathcal{f} : \text{Y} \to \text{X} \times_{RZ_k(K)} RZ_k(K') \) is injective (resp. bijective).

As an absolute version of the last definition, we note that \((X, K, \phi)\) is separated (resp. proper) if \( \phi \) is injective (resp. bijective).

Theorem 3. A morphism \( Y \to X \) of \( k \)-analytic spaces is separated (resp. boundaryless) at \( y \in Y \) iff \( \mathcal{f} : \text{(Y, y)} \to \text{(X, x)} \) is separated (resp. proper).

(Recall that for an affinoid space \( X = M(A) \), \( x \in \text{Int}(X) \) iff \( \text{red}(x) \in \text{Max}(\hat{A}) \) is a closed point.)

Corollary 1. \( f : Y \to X \) is proper iff \( f \) is topologically proper, and \( \mathcal{f}_y : \text{(Y, y)} \to \text{(X, f(y))} \) is proper for all \( y \in Y \).

Example 1. Let \( \mathbb{B}(0, 1) = \{|z| \leq 1\} = M(k(T)) \). Then \( \text{red}(\mathbb{B}(0, 1)) \cong \text{Spec}(k[T]) \cong \mathbb{A}_k^1 \).

If \( x \) is a type II point, \( x \neq \zeta_{\text{Gauss}} \), then \( x \in \text{Int}(X) \), and \( \text{(X, x)} \cong \mathbb{P}^1_k \), the entire Riemann-Zariski space. On the other hand, if \( x = \zeta_{\text{Gauss}} \), then \( x \notin \text{Int}(X) \), and \( \text{(X, x)} \cong \mathbb{A}_k^1 \), which is not the entire Riemann-Zariski space.
1 Formal schemes

Let $k$ be a nontrivially vauled complete non-archimedean field. All $k$-analytic spaces today are strict. Let $R = k^\circ$, and fix $\pi \in R$ with $0 < |\pi| < 1$. We have an algebra

$$R(x_1, \ldots, x_n) := \left\{ \sum_{I \in \mathbb{N}^n} a_I X^I \mid |a_i| \leq 1 \forall I \text{ and } |a_I| \to 0 \text{ as } \|I\| \to \infty \right\}.$$ 

A topologically finitely presented $R$-algebra is an $R$-algebra $A$ of the form

$$A = R(x_1, \ldots, x_n)/I$$

with $I = (f_1, \ldots, f_n)$ a finitely generated ideal. An admissible $R$-algebra is a topologically finitely presented algebra which is flat over $R$ (equivalently, $\pi$-torsion free). If $A$ is a topologically finitely presented $R$-algebra, we set $\overline{X} = \text{Spec } (A/\pi A)$ as a topological space. This is a finite type scheme over $\overline{k}$. For $f \in A$, we set

$$\overline{X}_f = \{ x \in \overline{X} : f(x) \neq 0 \},$$

and let $A_{\{f\}} = A(\overline{X})/(1 - fx)$.

Exercise 1. The ring $A_{\{f\}}$ depends only on $\overline{X}_f$ and not on $f$.

Theorem 1. The assignment $\overline{X}_f \sim A_{\{f\}}$ uniquely extends to a sheaf $\mathcal{O}_A$ of $R$-algebras on $\overline{X}$. The stalks are local rings.

Definition 1. A (topologically finitely presented) affine formal scheme $\text{Spf}(A)$ over $R$ is a locally ringed space of the form $(\overline{X}, \mathcal{O}_A)$. A (topologically finitely presented) formal scheme over $R$ is a locally ringed space $(\overline{X}, \mathcal{O}_X)$ with $\mathcal{O}_X$ a sheaf of $R$-algebras, which is locally isomorphic (over $R$) to an affine formal $R$-scheme.

2 Raynaud’s generic fiber functor

We have a functor

$$\text{Spf}(A) \rightsquigarrow \text{Sp}(A \otimes_R k) \text{ or } \mathcal{M}(A \otimes_R k),$$

which carries Zariski open subsets to quasi-compact admissible opens in $\text{Sp}(A)$ and compact special sets in $\mathcal{M}(A \otimes_R k)$. This construction extends canonically to a functor

$$\{ \text{topologically finitely presented formal } R\text{-schemes} \} \to \{ \text{rigid spaces over } k \}$$

or $\to \{ \text{k-analytic spaces} \}$

defined by $\mathfrak{X} \rightsquigarrow \mathfrak{X}_k$ or $\mathfrak{X}^{an}$. 

1
Proposition 1. This functor is faithful on the admissible formal $R$-schemes.

Given a formal scheme $\mathfrak{X}$, we have specialization (reduction) maps $sp: \mathfrak{X}_K \to \overline{\mathfrak{X}}$ where $\overline{\mathfrak{X}}$ is the scheme over $\overline{K}$ underlying $\mathfrak{X}$, and $sp: \mathfrak{X}_{\text{an}} \to \overline{\mathfrak{X}}$. The second map is defined by the following: for $x \in \mathfrak{X}_{\text{an}}$, there is a map $\chi_x: M(\mathcal{H}(x)) \to \mathfrak{X}_{\text{an}}$. This gives rise to $\chi_x^\circ: \text{Spf}(\mathcal{H}(x)^\circ) \to \mathfrak{X}$, so we get $\overline{\mathfrak{X}}_x = \text{Spec}(\mathcal{H}(x)) \to \overline{\mathfrak{X}}$.

Example 1. If $A_k$ is a reduced affinoid algebra, then $A_k^\circ$ is an admissible $R$-algebra. However, when $X$ is a gluing of $A_{\text{an}}$ and $B_{\text{an}}$ along $U_{\text{an}}$, it may be impossible to glue $\text{Spf}(A_k^\circ)$ and $\text{Spf}(B_k^\circ)$ along $\text{Spf}(U_k^\circ)$. Raynaud solved this problem by using admissible formal blowing-ups.

Theorem 2. (Raynaud) Every paracompact Hausdorff $k$-analytic space $X$ admits an admissible formal model, i.e., an admissible formal scheme $\mathfrak{X}$ such that $\mathfrak{X}_{\text{an}} \cong X$, and every morphism of such space is induced by a morphism of admissible formal models.

3 Temkin’s reduction functor

Let $X$ be a strict $k$-analytic space, $x \in X$, $X'$ a compact neighborhood of $x$, and $\mathfrak{X}'$ a formal model for $X'$. Define $(\tilde{X}, x)$ to be the class of $\text{Spec}(\mathcal{H}(x)) \to (\mathfrak{X}', sp(x))$ in Bir$_k$.

Fact 1. This only depends on $(X, x)$, not on $X', \mathfrak{X}'$, and gives a functor. This functor agrees with the reduction functor which was defined previously using adic spaces.

Definition 2. A scheme model for an object (morphism) in bir$_k$ is an object (morphism) of Var$_k$ in the corresponding class.

Theorem 3. (Temkin) 1) A birational space is affine if and only if it admits an affine scheme model.

2) A morphism of birational spaces is separated (resp. proper) if and only if it admits a separated (resp. proper) scheme model if and only if every scheme model is separated (resp. proper).

Theorem 4. (Temkin, Conrad) Let $X, \mathfrak{X}'$, $x \in X$ as before. The following are equivalent:

1) $x$ is a good point,

2) $(\tilde{X}, x)$ is affine (i.e., has an affine scheme model),

3) The normalization $S$ of the Zariski closure of $sp(x)$ in $\mathfrak{X}'$ is proper over an affine $\overline{k}$-scheme of finite type,

4) $\Gamma(S, \mathcal{O}_S)$ is finitely generated over $\overline{k}$ and $S \to \text{Spec}(\Gamma(S, \mathcal{O}_S))$ is proper.

Corollary 1. If $S$ is quasi-affine but not affine, then $x$ is not a good point.

Example 2. Let $X = \mathbb{B}_k^2/(\mathbb{B}_k^2)^-$, $\zeta$ Gauss point of $X \subset \mathbb{B}_k^2$, $\mathfrak{X}$ the complement of 0 in Spf$(R(t, t'))$. Then $\mathfrak{X} = \mathbb{A}_k^2 - \{0\}$ is quasi-affine but not affine, and $sp(\zeta)$ is the generic point of $\mathfrak{X}$. By 4) $\to$ 1), $\zeta$ is not good.
Math 274: Non-Archimedean Geometry

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Structure of analytic curves (following [BPR, §5])

Assume for simplicity that \( k \) is an algebraically closed, non-trivially valued field. Let \( X \) be a smooth connected algebraic curve over \( k \), not necessarily proper, and let \( \tilde{X} \) be the (unique) smooth proper curve birational to \( X \). Let \( D = \tilde{X} \setminus X \), the set of “punctures” or “points at infinity” (so \( D \) is a finite set of closed points).

**Theorem 1.** There exists a finite set \( V \) of type 2 points in \( X^{an} \) such that \( \tilde{X}^{an} \setminus V \) is a disjoint union of open balls and finitely many open annuli, such that each point of \( D \) is contained in a distinct such open ball. Equivalently, \( X^{an} \setminus V \) is a disjoint union of open balls, finitely many open annuli, and finitely many punctured open balls, with the punctures corresponding bijectively to the points of \( D \).

**Terminology:** We call such a set \( V \) a semistable vertex set of \( X \).

**Idea of proof:** By the semistable reduction of Bosch and Lütkebohmert, there exists a semistable (formal) model \( \mathcal{X}/R \) for \( \tilde{X}^{an} \), i.e. an (admissible) formal scheme \( \mathcal{X}/R \) with Raynaud generic fiber \( \tilde{X}^{an} \) and special fiber \( X^{s} \) a semistable curve over \( \bar{k} \) (a reduced separated one-dimensional scheme of finite type over \( \bar{k} \) whose only singularities are ordinary double points). We have the reduction map \( \text{red} : \tilde{X}^{an} \to X^{s} \); recall it is surjective and anticontinuous. From Berkovich and Bosch/Lütkebohmert, we have:

- If \( \zeta \) is a generic point of \( X^{s} \), then \( \text{red}^{-1}(\zeta) = \{\text{pt}\} \).
- If \( z \in X^{s} \) is a smooth point, then \( \text{red}^{-1}(z) \) is isomorphic as a \( k \)-analytic space to the open unit ball \( B(1) \) (a similar fact holds in arbitrary dimensions).
- If \( z \in X^{s} \) is an ordinary double point, then \( \text{red}^{-1}(z) \) is isomorphic as a \( k \)-analytic space to an open annulus.

**Remark 1.** By suitable admissible formal blowups, one can arrange that all points of \( D \) reduce to distinct smooth points.

**Remark 2.** One can also prove the above structure theorem for analytic curves without using the semistable reduction theorem, and then deduce the semistable reduction theorem as a result (cf. Temkin, Ducros, Thuiller).

Recall that any open annulus \( A \) is isomorphic to a standard open annulus

\[ \{z \in A^{1}_{\text{Berk}} : r < |z| < 1\} \]

for a unique \( r < 1 \). We call \(- \log(r)\) the modulus of \( A \).
Under such an isomorphism, the set
\[ \{ z = \zeta_{0, \alpha} : r < \alpha < 1 \} \]
is intrinsic to the annulus $A$; we call it the skeleton $\Sigma(A)$ of $A$. We can parametrize it by $-\log(\alpha)$, and this allows us to identify $\Sigma(A)$ with the open interval $(0, -\log(r))$.

For the punctured ball $B = \{ z \in \mathbb{A}^1_{\text{Berk}} : 0 < |z| < 1 \}$, we define the skeleton of $B$ to be
\[ \Sigma(B) = \{ \zeta_{0, \alpha} : 0 < \alpha < 1 \}. \]
Via the parameter $-\log(\alpha)$, we may identify $\Sigma(B)$ with the open ray $(0, \infty)$.

The skeleton $\Sigma = \Sigma(X^{\text{an}}, V)$ of $V$ is the metric graph obtained as
\[ V \cup \left( \bigsqcup_A \Sigma(A) \right) \cup \left( \bigsqcup_B \Sigma(B) \right) \subset X^{\text{an}}, \]
with the disjoint unions taken over the open annuli $A$ and punctured open balls $B$ of the structure theorem for analytic curves.

**Fact:** $\Sigma$ is connected.

**Example 1.** $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $V = \{ \zeta_{\text{Gauss}} \}$.

![Diagram](example1.png)

**Example 2.** $X = E$ an elliptic curve.

**Case 1:** $E$ has good reduction.

\[ \Sigma = \bullet \text{ (point)} \]

**Case 2:** $E$ has bad reduction.

\[ \Sigma = \text{circle with length} = \log |j_E| \]

**Example 3.** $E$ an elliptic curve with bad reduction, $X = E \setminus E[2]$.

**Case 1:** $k$ has residue characteristic $\neq 2$.

![Diagram](example3.png)
Case 2: $k$ has residue characteristic 2.

Proposition 1. Let $X, V$ be as above. Then

$$\Sigma(X, V) = \{ x \in X^{an} : x \text{ does not have an affinoid neighborhood isomorphic to } \mathbb{B}_k^1 \text{ and disjoint from } V \}.$$
1 Retraction to the Skeleton

Let $X/k$ be a smooth algebraic curve as in the previous lecture. Let $V$ be a semistable vertex set of $X^\text{an}$. Let $\Sigma$ denote the skeleton $\Sigma(X^\text{an}, V)$. We can define a retraction

$$\tau_V : X^\text{an} \to \Sigma$$

as follows. For $x \in X^\text{an} \setminus \Sigma$, there exists a unique connected component $B_x$ of $X^\text{an} \setminus \Sigma$ containing $x$, with $B_x$ isomorphic to an open ball such that $\partial B_x \subset \Sigma$. Then we define

$$\tau_V(x) := \partial B_x.$$

**Theorem 1** (Berkovich). The map $\tau_V$ is continuous and induces a homotopy equivalence.

**Theorem 2.** The natural map

$$X^\text{an} \to \lim_{\overset{\longrightarrow}{V}} \Sigma(X^\text{an}, V)$$

is a homeomorphism, where the limit is over semistable vertex sets $V$.

**Definition 1.** For $x \in X^\text{an}$, set $g(x) = 0$ if $x$ is not of Type II, and if $x$ is of Type II, set $g(x)$ to be the genus of the unique nonsingular projective curve over $\hat{k}$ with function field $\mathcal{H}(x)$.

**Theorem 3** (Genus formula).

$$g(\hat{X}) = g(\Sigma(\hat{X}^\text{an}, V)) + \sum_{x \in X^\text{an}} g(x),$$

where $g(\Sigma(\hat{X}^\text{an}, V))$ is just $\dim_{\mathbb{R}} H_1(\Sigma, \mathbb{R})$.

**Example 1.** If $g(\hat{X}) = 1$, there are two possibilities: either the genus comes from a Type II point, or it comes from genus in the skeleton. These possibilities are pictured in Figure 1.

**Definition 2.** The *Euler characteristic* of $X$ is

$$\chi(X) = 2 - 2g(\hat{X}) - \#D,$$

where $D = \hat{X} \setminus X$.

**Definition 3.** A semistable vertex set $V$ is *stable* if there’s no $x \in V$ with $g(x) = 0$ and $\deg_{\Sigma}(x) < 3$.

**Theorem 4** (Stable Reduction Theorem). If $\chi(x) < 0$, then there exists a unique stable vertex set for $X$. If $\chi(x) \leq 0$, then there exists a unique set-theoretically minimal skeleton of $X$. 


Example 2. First we’ll consider the case of \( g = 0 \). If \( D = \hat{X} \setminus X \) has only two points, then \( \chi(X) = 0 \) and we’re in the weaker case of having a unique set-theoretically minimal skeleton of \( X \), rather than a unique stable vertex set. The minimal skeleton in this case is the line segment connecting the two points of \( D \), say \( 0 \) and \( \infty \), which is not a stable vertex set. However, if \( D \) has at least three points, then the stable reduction theorem says there will be a stable vertex set.

Now consider \( g = 1 \), so we’re considering \( X \) an elliptic curve. As argued in the previous lecture, in the case of good reduction, the skeleton is simply a point. However, for bad reduction, the skeleton \( \Sigma \) is a loop, and the minimal semistable vertex set is any Type II point on \( \Sigma \).

Once \( g \geq 2 \), there’s no need to mark any points whatsoever, since we’re guaranteed to have \( \chi(\hat{X}) < 0 \) regardless of how many points are in \( D \).

2 Non-Archimedean Picard Theorems

Back in the Archimedean world, we have the following theorems.

Theorem 5 (Picard’s Little Theorem). A nonconstant entire function on \( \mathbb{C} \) omits at most one complex value.

Theorem 6 (Picard’s Moderately-Sized Theorem). Let \( X \) be a Riemann surface with \( \chi(X) < 0 \). Then any analytic map \( \mathbb{C} \rightarrow X \) is constant.

Proof. Since \( \chi(X) < 0 \), the analytic universal cover \( \Omega \) of \( X \) is the open unit disk. By the standard properties of universal covers, our map \( \mathbb{C} \rightarrow X \) lifts as pictured:

\[
\begin{array}{c}
\Omega \\
\downarrow \\
\mathbb{C} \\
\rightarrow X
\end{array}
\]

The map \( \mathbb{C} \rightarrow \Omega \) is a bounded analytic function since \( \Omega \) is the unit disk, and is thus constant, implying that the map \( \mathbb{C} \rightarrow X \) is constant as well. \( \square \)
Berkovich proved a non-Archimedean analogue of Theorem 6.

**Theorem 7** (Berkovich). Let $k$ be a complete, nontrivially valued non-Archimedean field. Any analytic map $f : \mathbb{A}^1 \to X$ where $X$ is a (smooth) analytic curve with $\chi(X) \leq 0$ is constant. (Here the $\chi(X) \leq 0$ is *not* strict, unlike in the Archimedean case.)

To prove this, we’ll need a definition and some facts.

**Definition 4.** We say that $X$ is *totally degenerate* if $g(X) = g(\Sigma)$, that is, $g(x) = 0$ for all $x \in X$.

**Facts:**

1. $X$ has a topological (and analytic) universal cover $\Omega$. (In particular, $\Omega$ is a topological cover endowed with a unique analytic structure such that any map from a simply connected space to $X$ factors through $\Omega$.)

2. (Due to Tate.) If $g(X) = 1$ and $X$ is totally degenerate, then $\Omega \cong \mathbb{C}_m = (\mathbb{P}^1)^{an} \setminus \{0, \infty\}$.

3. (Due to Mumford.) If $g(X) \geq 2$ and $X$ is totally degenerate, then $\Omega \cong (\mathbb{P}^1)^{an} \setminus L$,

   where $L \subset \mathbb{P}^1(k)$ is a nonempty perfect set.

**Proof of Berkovich’s Theorem.** Without loss of generality, we’ll assume $k = \overline{k}$.

We now split into three cases.

- First off, if $g(\hat{X}) = 0$, then all we need to show is that an entire nonconstant function $\mathbb{A}^1 \to \mathbb{A}^1$ must be surjective. This may be shown using Newton polygons.

- If $\hat{X}^{an}$ is totally degenerate, then $\Omega \subset (\mathbb{P}^1)^{an} \setminus L$ where $L \subset \mathbb{P}^1(k)$, with $|L| \geq 2$. This means we may view $\Omega \subset (\mathbb{A}^1)^{an} \setminus \{0\}$. By the universal property of $\Omega$, we obtain the following diagram:

  \[
  \begin{array}{cccc}
  & & \Omega \\
  (\mathbb{A}^1)^{an} \ar[u] \ar[r] & X \\
  \end{array}
  \]

  The map $(\mathbb{A}^1)^{an} \to \Omega \hookrightarrow (\mathbb{A}^1)^{an} \setminus \{0\} \twoheadrightarrow (\mathbb{A}^1)^{an}$ is not surjective and so must be constant (by the first case). Hence the map $(\mathbb{A}^1)^{an} \to X^{an}$ is constant.

- If we’re not in one of the above cases, then there exists some $x \in X^{an}$ with $g(x) \geq 1$. This means that $x$ is not in the image of $f$, since if $f(y) = x$, we would have an injection $\overline{H}(x) \hookrightarrow \overline{H}(y)$ of a function field of genus $\geq 1$ into a function field of genus $0$, which is impossible.

Let $U_1, U_2$ be distinct open balls in $X^{an} \setminus \{x\}$. Now, $f((\mathbb{A}^1)^{an})$ is path connected, but any path from a point of $U_1$ to a point of $U_2$ must go through $x$. Hence some $U_i = U$ is disjoint from $\text{im}(f)$.

Choose $y \in U$ in $X(k)$. By Riemann-Roch, we may pick a rational function $h$ with poles only at $y$.

Consider the bounded map $h \circ f : \mathbb{A}^1 \to X^{an} \setminus U \to \mathbb{P}^1 \setminus \{\infty\} \cong \mathbb{A}^1$. It follows that $h \circ f$ is constant, implying in turn that $f$ is constant.