CALCULUS PROBLEMS

1. Let \( a_1, \ldots, a_k \) be nonnegative real numbers. Evaluate
\[
\lim_{n \to \infty} \left( a_1^n + \cdots + a_k^n \right)^{1/n}.
\]

2. a. Find all differentiable functions \( f : (0, \infty) \to \mathbb{R} \) such that \( f(xy) = f(x) + f(y) \) for all \( x, y > 0 \).

   b. Find all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(x + y) = f(x) + f(y) \) for all \( x, y \in \mathbb{R} \).

3. Let \( a \) be a positive real number, and define a sequence \( x_n \) by \( x_0 = 0 \) and \( x_{n+1} = x_n^2 + a \) for \( n \geq 0 \). For which values of \( a \) does \( \lim_{n \to \infty} x_n \) exist?

4. Show that for all integers \( n > 1 \),
\[
\frac{1}{2ne} < \frac{1}{e} - \left( 1 - \frac{1}{n} \right)^n < \frac{1}{ne}.
\]

5. Let \( f(x) \) be differentiable on \([0, 1]\) with \( f(0) = 0 \) and \( f(1) = 1 \). For each positive integer \( n \), show that there exist distinct points \( x_1, x_2, \ldots, x_n \) in \([0, 1]\) such that
\[
\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n.
\]

6. For each \( x > e^e \), define a sequence \( S_x = u_0, u_1, u_2, \ldots \) recursively as follows: \( u_0 = e \), while for \( n \geq 0 \),
\[
u_{n+1} = \log_{u_n} x.
\]
Prove that \( S_x \) converges to a real number \( g(x) \), and that the function \( g \) defined in this way is continuous for \( x > e^e \).