Unlikely intersections in complex dynamics

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(Joint work with L. DeMarco)

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The Masser-Zannier theorem

Theorem (Masser–Zannier)

Let $P_\lambda, Q_\lambda$ be linearly independent points in $E_\lambda(\overline{\mathbb{C}(\lambda)})$, where $E_\lambda : y^2 = x(x - 1)(x - \lambda)$ is the Legendre family of elliptic curves. Then

$$\{ \lambda \in \mathbb{C} \mid P_\lambda \text{ and } Q_\lambda \text{ are both torsion} \}$$

is finite.
Motivated by his joint work with Masser, and by the analogy between torsion points on elliptic curves and preperiodic points for dynamical systems, Zannier posed the following question in 2008:

**Question**: What can be said about the set of \( c \in \mathbb{C} \) such that both 0 and 1 are preperiodic for the map \( z \mapsto z^2 + c \)?

- \( c = 0 \): \( 0 \mapsto 0, 1 \mapsto 1 \)
- \( c = -1 \): \( 1 \mapsto 0 \mapsto -1 \mapsto 0 \)
- \( c = -2 \): \( 0 \mapsto -2 \mapsto 2 \mapsto 2, 1 \mapsto -1 \mapsto -1 \).

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**Open problem:** Are there any other such complex parameters $c$?
Main theorem of [BDM1]

Theorem (B.-DeMarco)

Fix $a \neq b$ in $\mathbb{C}$ with $a \neq \pm b$. Then

$$\{ c \in \mathbb{C} \mid a \text{ and } b \text{ are both preperiodic for } z^2 + c \}$$

is finite.
A rational map $f \in \mathbb{C}(t)$ is called **postcritically finite** (PCF) if the critical points of $f$ are all preperiodic.

PCF maps are very important in complex dynamics, roughly speaking because of the

**Slogan:** The dynamics of a rational map $f$ under iteration is governed by what happens to the critical points.

Here is a plot of some values of $c \in \mathbb{C}$ for which $z^2 + c$ is PCF:
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Here is a plot of some values of \( c \in \mathbb{C} \) for which \( z^2 + c \) is PCF:
Let $\text{Rat}_d \subset \mathbb{P}^{2d+1}$ be the space of rational maps of degree $d$, and let $M_d = \text{Rat}_d/\text{PSL}_2(\mathbb{C})$. Let $P_d$ be the subset of $M_d$ consisting of (conjugacy classes of) \textbf{polynomial} maps.

\textbf{Fact:} The PCF maps form a countable, Zariski dense subset of $P_d$. They form a Zariski dense subset of $\text{Rat}_d$ which is countable outside of the \textbf{Lattès locus} $\text{Lat}_d$ (which is either empty, if $d$ is not a square, or an algebraic curve).

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There are many reasons to subscribe to the idea that PCF maps are analogous to CM points; for example:

- The set of PCF points in $\text{Rat}_d \setminus \text{Lat}_d$ has bounded height and thus there are only finitely many non-Lattès PCF maps defined over number fields of bounded degree (analogue of Gauss’s class number problem). [Benedetto-Ingram-Jones-Levy]

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By analogy with the André-Oort conjecture, it is natural to ask which algebraic subvarieties of $M_d$ contain a dense set of “special” (PCF) points. Before formulating a conjectural answer, let’s look at some examples.

**Example 1:** Consider the parametrized curve in $P_3$ given by

$$f_t(z) = z^3 - 3t^2z + i.$$ 

The (finite) critical points are $c_1(t) = t$ and $c_2(t) = -t$, which are “dynamically independent” in a sense to be made precise shortly. It follows from the main theorem of [BDM2] that there are only a finite number of PCF maps in this family of cubic polynomials.
By analogy with the André-Oort conjecture, it is natural to ask which algebraic subvarieties of $M_d$ contain a dense set of “special” (PCF) points. Before formulating a conjectural answer, let’s look at some examples.

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Example 2: For the curve in $M_4$ defined by the Lattès family
\[ f_\lambda(z) = \frac{(z^2 - \lambda)^2}{4z(z - 1)(z - \lambda)}, \]
every point is PCF. By a (difficult) theorem of Thurston, Lattès curves are the only examples of positive-dimensional subvarieties of $M_d$ consisting entirely of PCF points (which one might call very special).
Example of a special subvariety

**Example 3:** Consider the parametrized curve in $P_5$ given by

$$f_t(z) = z^2(z^3 - t^3).$$

One can show that this curve is special, i.e., the above family contains infinitely many PCF maps. To see this, let $\beta = \frac{3\sqrt{2}}{5}$ and let $\omega$ be a primitive cube root of unity. Then the critical points are $c_j(t) = \omega^j \beta t$ for $j = 1, 2, 3$ and $c_4(t) = 0$.

The critical point $c_4$ is persistently periodic, since it’s fixed in every fiber. And for $i, j \in \{1, 2, 3\}$ and $t_0 \in \mathbb{C}$, we have

$$c_i(t_0) \text{ is preperiodic if and only if } c_j(t_0) \text{ is preperiodic.}$$

Indeed, these three points satisfy the critical orbit relations

$$f_t^{(2)}(c_j(t)) = \omega \cdot f_t^{(2)}(c_{j-1}(t))$$

and in addition we have

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**Motivation from arithmetic geometry**

**Complex dynamics**

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**Baker-DeMarco**

**Unlikely intersections in complex dynamics**
Dynamically dependent orbits

Let $V$ be an irreducible complex algebraic variety of dimension at least $1$, and let

$$f = \{ f_t : t \in V \}$$

be a family of rational maps of degree at least $2$ over $\mathbb{C}$.

The dimension of the family is the dimension of the image of $V$ in $M_d = \text{Rat}_d/\text{PSL}_2(\mathbb{C})$ under the morphism $V \to \text{Rat}_d$ defined by $f$.

Let $a_1(t), \ldots, a_m(t) \in \mathbb{P}^1(\mathbb{C}(V))$ be marked points. We say that $a_{i_1}, \ldots, a_{i_n}$ have dynamically dependent orbits if the point $(a_{i_1}, \ldots, a_{i_n}) \in \mathbb{A}^n(\mathbb{C}(V))$ is contained in an algebraic hypersurface which is invariant under the action of $(f, f, \ldots, f)$. 
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Conjecture (B.-DeMarco): Let \( f = \{ f_t : t \in V \} \) be an \( N \)-dimensional family of rational maps of degree at least 2 over \( \mathbb{C} \), and let \( a_1(t), \ldots, a_m(t) \in \mathbb{P}^1(\mathbb{C}(V)) \) be marked points. Then the set of \( t \in V \) such that \( a_1(t), \ldots, a_m(t) \) are simultaneously preperiodic is Zariski dense in \( V \) if and only if at most \( N \) of the marked points \( a_1, \ldots, a_m \) have dynamically independent orbits.

We call the special case where \( a_1(t), \ldots, a_m(t) \) are the critical points of \( f \) the dynamical André-Oort conjecture, since it gives a conjectural characterization of the special subvarieties.
The general conjecture of [BDM2]

**Conjecture (B.-DeMarco):** Let $f = \{f_t : t \in V\}$ be an $N$-dimensional family of rational maps of degree at least 2 over $\mathbb{C}$, and let $a_1(t), \ldots, a_m(t) \in \mathbb{P}^1(\mathbb{C}(V))$ be marked points. Then the set of $t \in V$ such that $a_1(t), \ldots, a_m(t)$ are simultaneously preperiodic is Zariski dense in $V$ if and only if at most $N$ of the marked points $a_1, \ldots, a_m$ have dynamically independent orbits.

We call the special case where $a_1(t), \ldots, a_m(t)$ are the critical points of $f$ the **dynamical André-Oort conjecture**, since it gives a conjectural characterization of the special subvarieties.
As a concrete special case, one has:

**Conjecture:** Let $C \subset \mathbb{A}^2_C$ be an algebraic curve containing infinitely many points $(a, b)$ such that both $z^2 + a$ and $z^2 + b$ are PCF. Then $C$ is either horizontal, vertical, or diagonal.
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The one-parameter polynomial case

We proved the following stronger version of the main conjecture for 1-parameter families of polynomial maps assuming that the \( a_i(t) \) are also polynomials:

**Theorem (B.-DeMarco)**

Let \( f = \{f_t\} \) be a one-parameter family of complex polynomials of degree at least 2, and let \( a_1(t), \ldots, a_m(t) \in \mathbb{C}[t] \) be polynomials. Then the set of \( t \in V \) such that \( a_1(t), \ldots, a_m(t) \) are simultaneously preperiodic is infinite if and only if every pair of critical points which are not persistently periodic satisfies a critical orbit relation of the form

\[
f^{(m)}(c_i) = h \circ f^{(n)}(c_j)
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where \( h \) is a polynomial that commutes with some iterate of \( f \).
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where $h$ is a polynomial that commutes with some iterate of $f$. 
Ingredients in the proof

The proof involves several different kinds of ingredients, notably:

- An adelic equidistribution theorem for Galois orbits of points of small dynamical height
- Potential theory on both $\mathbb{P}^1(\mathbb{C})$ and its non-Archimedean Berkovich space analogue
- Complex analysis, especially univalent function theory and the theory of normal families
- The recent work of Medvedev-Scanlon, who use the classical methods of Ritt to classify the invariant subvarieties for a certain class of polynomial dynamical systems.
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For some non-polynomial examples, consider the curve $\text{Per}_1(\lambda)$ in $P_3$ (resp. $M_2$) consisting of (conjugacy classes of) rational maps with a marked fixed point of multiplier $\lambda$.

**Theorem (B.-DeMarco for $P_3$, DeMarco-Wan-Ye for $M_2$)**

The curve $\text{Per}_1(\lambda)$ contains infinitely many PCF points if and only if $\lambda = 0$.

Note that even the case of $P_3$ is not a consequence of the previous theorem, since the critical points in this family are not parametrized by polynomials.
The Milnor curves

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The Masser-Zannier theorem revisited

Recently, DeMarco, Wan, and Ye have given a new proof of the Masser-Zannier theorem using the methods of proof from [BDM1] and [BDM2], as applied to the degree-four Lattès family. The method of proof in fact yields a Bogomolov-type strengthening of the Masser-Zannier theorem which applies to points of small height and not just torsion points:

**Theorem (DeMarco-Wan-Ye)**

For $a \neq b$ in $\mathbb{Q}\setminus\{0, 1\}$, there exists $\epsilon > 0$ such that the set of $\lambda \in \mathbb{Q}\setminus\{0, 1\}$ for which the points $P_\lambda = (a, \sqrt{a(a-1)(a-\lambda)})$ and $Q_\lambda = (b, \sqrt{b(b-1)(b-\lambda)})$ both have Néron-Tate canonical height less than $\epsilon$ is finite.
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A theorem of Ghioca-Krieger-Nguyen

**Theorem (Ghioca-Krieger-Nguyen)**

Let \( f(z) \in \mathbb{C}[z] \) be a non-constant polynomial. There are infinitely many \( t \in \mathbb{C} \) such that both \( z^d + t \) and \( z^d + f(t) \) are PCF if and only if \( f(z) = \zeta z \) for some \((d - 1)\text{st}\) root of unity \( \zeta \).

For \( d = 2 \), this is the special case \( y = f(x) \) of the conjecture than an algebraic curve in \( \mathbb{A}^2 \) containing infinitely many points \((a, b)\) such that both \( z^2 + a \) and \( z^2 + b \) are PCF if and only if \( C \) is either horizontal, vertical, or diagonal.
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Thank you!

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